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Show all pertinent work for full credit.

1. (8.2c Complex Roots: 10 pts) Consider the system $\vec{x}'(t) = \begin{pmatrix} -1 & -2 \\ 8 & -1 \end{pmatrix} \vec{x}(t)$

(a) Find the Characteristic Equation. Show your work.

$$\begin{vmatrix} -1-\lambda & -2 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

3

$$(\lambda+1)(\lambda+1) - 8(-2) = 0$$

$$\lambda^2 + 2\lambda + 1 + 16 = 0 \rightarrow \boxed{\lambda^2 + 2\lambda + 17 = 0}$$

Characteristic Equation: $\boxed{1\ \lambda^2 + 2\ \lambda + 17 = 0}$

Find the complex roots via the quadratic equation:

$$\lambda = \frac{-(2) \pm \sqrt{2^2 - 4(1)(17)}}{2 \cdot 1}$$

3

$$= \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm \frac{\sqrt{-64}}{2} = -1 \pm \frac{8i}{2} = -1 \pm 4i$$

$\boxed{\lambda_1 = -1 + 4i}$ and $\boxed{\lambda_2 = -1 - 4i}$

(b) This system here $\vec{x}'(t) = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \vec{x}(t)$ has an eigenpair of $\lambda_1 = 2i$, $\vec{\eta}_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix}$

Using the formula in the last pages state the general solution to the system with c_1 and c_2 coefficients (use class or text notation). Put answer in matrix/vector format.

$$\lambda = 0 \pm 2i \qquad \alpha = 0, \beta = 2, \quad \vec{n} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\vec{a}} \pm \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\vec{b}} i$$

4

- $\vec{x}_1(t) = e^{0 \cdot t} \cos(2t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} - e^{0 \cdot t} \sin(2t) \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
- $\vec{x}_2(t) = e^{0 \cdot t} \sin(2t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + e^{0 \cdot t} \cos(2t) \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ -\sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(2t) + 2\cos(2t) \\ -\sin(2t) \end{pmatrix}}$$

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2. (8.2b Double Roots: 10 pts) The system $\vec{x}'(t) = \begin{pmatrix} 11 & -25 \\ 4 & -9 \end{pmatrix} \vec{x}(t)$ has repeated eigenvalues of $\lambda_{1,2} = 1, 1$ and the only eigenvector is $\vec{\eta}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

a) Find the helper vector $\boldsymbol{\rho}$ that satisfies $(\mathbf{A} - \lambda \mathbf{I})\,\boldsymbol{\rho} = \boldsymbol{\eta}$

$$\left.\left(\begin{array}{cc|c} 11-\lambda & -25 & 5 \\ 4 & -9-\lambda & 2 \end{array}\right)\right|_{\lambda=1}$$

$$\left(\begin{array}{cc|c} 10 & -25 & 5 \\ 4 & -10 & 2 \end{array}\right) \begin{array}{l} \div 5 \\ \div 2 \end{array}$$

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$$\left(\begin{array}{cc|c} 2 & -5 & 1 \\ 2 & -5 & 1 \end{array}\right) \begin{array}{l} * -1 \\ \leftarrow \end{array}$$

$$\left(\begin{array}{cc|c} 2 & -5 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$2\rho_1 - 5\rho_2 = 1$

$\rho_1 = (1 + 5\rho_2)/2$

$$\vec{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{5\rho_2}{2} \\ \rho_2 \end{pmatrix}$$

$$\boxed{\vec{\rho} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}} \quad \text{if } \rho_2 = 0$$

b) State the general solution of the system using c_1 and c_2 as the leading coefficients. Use

$$\vec{x} = c_1 \mathbf{e}^{\lambda t} \vec{\eta} + c_2 \left(t\,\mathbf{e}^{\lambda t} \vec{\eta} + \mathbf{e}^{\lambda t} \vec{\rho} \right)$$

3

$$\vec{x}(t) = c_1 e^{1\cdot t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \left[t e^{1\cdot t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + e^{1\cdot t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right]$$

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \left[t e^{t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + e^{t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right]}$$

10

3. (7.1 ODE to Y(s): 10 pts) Using the definition of the Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

derive formula #2 in the Transform table provided in the Last Pages. Show all steps for full credit. Use the notation we used in class.

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt$$

5

$$= \int_0^\infty e^{at-st}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt \qquad \text{let} \begin{cases} u = (a-s)t \\ du = (a-s)\,dt \end{cases}$$

$$= \int_{t=0}^{t=\infty} \frac{e^u\,du}{a-s}$$

5

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_{t=0}^{\infty}$$

$$= \frac{1}{a-s}\left[\lim_{T\to\infty} e^{(a-s)T} - e^0\right] \qquad \to 0 \text{ if } a-s<0 \qquad \boxed{a<s}$$

$$= \frac{1}{a-s}\left[0-1\right]$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}} \qquad s>a$$

(10)

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Math 275 Su24, LACC, R.Erickson   *Test 4* (8.2c, 8.2b, 8.3, 7.1, 7.2)   Name: KEY

4. (8.3 Variation Params : 8 pts) Consider a nonhomogenous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

The following variation of parameter formula shows how to form a particular solution vector for this system

$$\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt.$$

where **X**(t) is made up of the solution vectors of the homogenous system of
$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$.

a) If the system $\vec{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\vec{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$ has homogeneous solution vectors $\vec{x}_1 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}$ form **X**(t).

← I know these are not the correct vectors ... FOLLOW THE DIRECTIONS

2

$$\boxed{\mathbf{X} = \begin{pmatrix} e^t & 3e^{-t} \\ e^t & e^{-t} \end{pmatrix}}$$

b) Find $\mathbf{X}^{-1}(t)$

3

$$\mathbf{X}^{-1} = \frac{\begin{pmatrix} e^{-t} & -3e^{-t} \\ -e^t & e^t \end{pmatrix}}{(e^t)(e^{-t}) - (e^t)(3e^{-t})} = \boxed{\begin{pmatrix} -e^{-t}/2 & \frac{3}{2}e^{-t} \\ e^t/2 & -\frac{e^t}{2} \end{pmatrix}}$$

$$1 - 3 = -2$$

$$= \frac{1}{2}\begin{pmatrix} -e^{-t} & 3e^{-t} \\ e^t & -e^t \end{pmatrix}$$

c) Now perform **X**(t) **f**(t)

Follow the directions

$$\begin{pmatrix} e^t & 3e^{-t} \\ e^t & e^{-t} \end{pmatrix}\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$$

3

$$= \boxed{\begin{pmatrix} e^{3t} + 3e^{-t} \\ e^{3t} + e^{-t} \end{pmatrix}}$$

8

was supposed to be $\mathbf{X}^{-1} f$ (Dyslexia @ 1am)

$$= \frac{1}{2}\begin{pmatrix} -e^{-t} & 3e^{-t} \\ e^t & -e^t \end{pmatrix}\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{-t}e^{2t} + 3e^{-t}e^{2t} \\ e^t \cdot e^{2t} - e^t e^{2t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2e^t \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} e^t \\ 0 \end{pmatrix}}$$

4. (continued 7 more pts) (d) Next integrate the results of (c) $\int X(t)f(t)dt$ was supposed to say

$\int \mathbb{X} f\,dt$

Follow the Directions

$$= \int \begin{pmatrix} e^{3t} + 3e^{-t} \\ e^{3t} + e^{-t} \end{pmatrix} dt$$

2

$$= \boxed{\begin{pmatrix} \frac{e^{3t}}{3} - 3e^{-t} \\ \frac{e^{3t}}{3} - e^{-t} \end{pmatrix}}$$

$\int \mathbb{X}^{-1} f\,dt$

$$= \int \begin{pmatrix} e^{t} \\ 0 \end{pmatrix} dt$$

$$= \boxed{\begin{pmatrix} e^{t} \\ 0 \end{pmatrix}}$$

(e) Finally form $\vec{x}_p(t) = X^{-1}\int X(t)f(t)dt$ Do it!

$$= \frac{1}{2}\begin{pmatrix} -e^{-t} & 3e^{-t} \\ e^{t} & -e^{t} \end{pmatrix}\begin{pmatrix} \frac{e^{3t}}{3} - 3e^{-t} \\ e^{3t}/3 - e^{-t} \end{pmatrix}$$

4

$$= \frac{1}{2}\begin{pmatrix} -e^{-t}\left[\frac{e^{3t}}{3} - 3e^{-t}\right] + 3e^{-t}\left[\frac{e^{3t}}{3} - e^{-t}\right] \\ e^{t}\left[\frac{e^{3t}}{3} - 3e^{-t}\right] - e^{t}\left[\frac{e^{3t}}{3} - e^{-t}\right] \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -\frac{1}{3}e^{2t} \cancel{+3e^{-2t}} + e^{2t} \cancel{-3e^{-2t}} \\ \cancel{\frac{1}{3}e^{4t}} - 3 \cancel{-\frac{1}{3}e^{4t}} + 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{2}{3}e^{2t} \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} e^{2t}/3 \\ -1 \end{pmatrix}}$$

$\vec{x}_p = \mathbb{X} \int \mathbb{X}^{-1} f\,dt$ { was supposed to say

$$= \begin{pmatrix} e^{t} & 3e^{-t} \\ e^{t} & e^{-t} \end{pmatrix}\begin{pmatrix} e^{t} \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}}$$

correct.

(f) State the general solution

$$\boxed{\vec{x}_g = \underbrace{c_1\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}}_{\vec{x}_c} + \underbrace{\begin{pmatrix} e^{2t}/3 \\ -1 \end{pmatrix}}_{\vec{x}_p}}$$

1

$$\boxed{\vec{x}_g = \underbrace{c_1\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}}_{\vec{x}_c} + \underbrace{\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}}_{\vec{x}_p}}$$

7)

6. (7.2 Y(s) to y(t): 15 pts) Solve via the Laplace Transform method (show all steps for full credit and use the notation and variables we use in class) Attach extra white paper if needed.

$$y'' - y' - 2y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = 0$$

(i) $\mathcal{L}[y''-y'-2y] = \mathcal{L}[0]$

$[s^2 Y - s y(0) - y'(0)] - [sY - y(0)] - 2Y = 0$

5

$[s^2 Y - s - 0] - [sY - 1] - 2Y = 0$

$(s^2 - s - 2)Y = s - 1$

(ii) $Y = \dfrac{s-1}{s^2-s-2} \Rightarrow \boxed{Y(s) = \dfrac{s-1}{(s-2)(s+1)}}$

$Y = \dfrac{A}{s-2} + \dfrac{B}{s+1} = \dfrac{A(s+1) + B(s-2)}{(s-2)(s+1)}$

5

Num: $As + Bs + A - 2B = s - 1$

$s^1: A + B = 1$ $\quad * -1$ $\Rightarrow$ $A + B = 1 \rightarrow \boxed{A = 1/3}$

$s^0: A - 2B = -1$ $\quad\quad\quad\quad\quad$ $-3B = -2 \rightarrow \boxed{B = 2/3}$

So $\boxed{Y(s) = \dfrac{1/3}{s-2} + \dfrac{2/3}{s+1}}$

(iii) $\mathcal{L}^{-1}$ $\quad\quad \mathcal{L}^{-1} \quad\quad \mathcal{L}^{-1}$

5

$\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$