30/30 + 30/30

Show all pertinent work for full credit. Closed book, closed internet, closed human helper, closed notes (there are notes on the last pages). Use the notation and methods we use in class or that are used in the text.

1. (4.6: 10 pts) Variation of parameters: Use formula 12 from Theorem 7 in the last pages to solve for the general solution of the ODE (budget your time and set up the integrals then come back to perform the integration time permitting):

$$y'' - 2y' + y = \underbrace{e^t / t}_{g(t)}$$

$p(t)$ $q(t)=1$

y_c: $y''-2y'+y=0$, $r^2-2r+1 \to (r-1)^2=0$ $\boxed{r=1,1}$

$$\boxed{y_c(t) = c_1e^t + c_2te^t}$$

$y_1 = e^t$, $y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = \boxed{e^{2t}}$$

$$v_1 = \int \frac{-g(t)y_2(t)}{W}\,dt \qquad v_2 = \int \frac{g(t)y_1}{W}\,dt$$

$$v_2 = -\int \frac{(e^t/t)(te^t)}{e^{2t}}\,dt \qquad v_2 = \int \frac{\left(\frac{e^t}{t}\right)e^t}{e^{2t}}\,dt$$

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$$v_1 = -t \qquad v_2 = \int \frac{1}{t}\,dt = \ln(t)$$

$$y = c_1y_1 + c_2y_2 + v_1y_1 + v_2y_2$$

$$\Rightarrow y = c_1e^t + c_2te^t + \underbrace{(-t)e^t}_{\text{combine}} + (\ln(t))te^t$$

$$\boxed{y = \underbrace{c_1e^t + c_2te^t}_{y_c} + \underbrace{te^t\ln(t)}_{y_p}}$$

2. (3.3: 10 pts) Read the passage below answer and the question that follows.

NETWORKS An electrical network having more than one loop also gives rise to simultaneous differential equations. As shown in Figure 3.3.3, the current $i_1(t)$ splits in the directions shown at point B_1, called a *branch point* of the network. By **Kirchhoff's first law** we can write

$$i_1(t) = i_2(t) + i_3(t). \tag{14}$$

We can also apply **Kirchhoff's second law** to each loop. For loop $A_1B_1B_2A_2A_1$, summing the voltage drops across each part of the loop gives

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2. \tag{15}$$

Similarly, for loop $A_1B_1C_1C_2B_2A_2A_1$ we find

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt}. \tag{16}$$

Using (14) to eliminate i_1 in (15) and (16) yields two linear first-order equations for the currents $i_2(t)$ and $i_3(t)$:

$$\begin{cases} L_1\dfrac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 = E(t) \\ L_2\dfrac{di_3}{dt} + R_1i_2 + R_1i_3 = E(t). \end{cases} \tag{17}$$

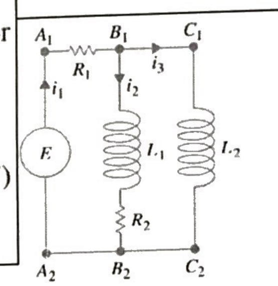

FIGURE 3.3.3 Network whose model is given in (17)

>> Write the first order ODE system (17) as a matrix/vector equation, of the form x'=Ax+g, defining the matrix and vectors.

$$\begin{cases} \dfrac{di_2}{dt} = \big(E(t) - (R_1+R_2)i_2 - R_1 \cdot i_3\big)/L_1 \\ \dfrac{di_3}{dt} = \big(E(t) - R_1i_2 - R_1i_3\big)/L_2 \end{cases}$$

So

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$$\boxed{\frac{d}{dt}\begin{pmatrix} i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} \dfrac{-(R_1+R_2)}{L_1} & \dfrac{-R_1}{L_1} \\ \dfrac{-R_1}{L_2} & -R_1/L_2 \end{pmatrix}\begin{pmatrix} i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} E/L_1 \\ E/L_2 \end{pmatrix}}$$

$$\vec{i}\,' = \underbrace{\qquad\qquad}_{A}\ \vec{i} + \vec{g}$$

 Name: KEY

3. (8.2a: 10 pts) Solve the IVP below given that the eigenvalues are $\lambda_{1,2} = 1,-1$ and their associated eigenvectors are listed besides the IVP below:

$$\vec{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \vec{x}(t) \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{\eta}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \vec{\eta}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a) State the general solution with c_1 and c_2 coefficients (use class or text notation)

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

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(b) Apply the initial condition to find the c_1 and c_2 coefficients

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \cdot 1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3c_1 + c_2 \\ c_1 + c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}}{(3\cdot 1)-(1\cdot 1)} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{\begin{pmatrix} 1\cdot 1 - 1\cdot 2 \\ -1\cdot 1 + 3\cdot 2 \end{pmatrix}}{2}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \Big/ 2 \Rightarrow \boxed{\begin{matrix} c_1 = -1/2 \\ c_2 = 5/2 \end{matrix}}$$

(c) State the specific solution

2

$$\boxed{\vec{x}(t) = -\frac{1}{2} e^{t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{5}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

or

$$\boxed{\begin{matrix} x_1(t) = -3/2\, e^{t} + 5/2\, e^{-t} \\ x_2(t) = -1/2\, e^{t} + 5/2\, e^{-t} \end{matrix}}$$

Math 275 Su24, LACC, R.Erickson   *Test 3* (4.6, B1, B2, EVP, 81., 8.2a)   Name: KEY

4. (8.2a: 10 pts) Sketch the phase-portrait for the system whose general solution is given below. Be sure to show directions on the trajectories you draw both on and off the eigenlines.

$$\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix}3\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

2 (a) Identify $\lambda_1 =$ 1 and $\lambda_2 =$ -1

2 (b) Fill in the eigenvectors: $\eta_1 = \begin{pmatrix}3\\1\end{pmatrix}$, $\eta_2 = \begin{pmatrix}1\\1\end{pmatrix}$

(c) Sketch the eigenlines and some trajectories off of the eigenlines. Show direction of motion with arrows on the eigenlines and the trajectories. ~~Plot the initial condition and its prospective trajectory.~~ Label pertinent items

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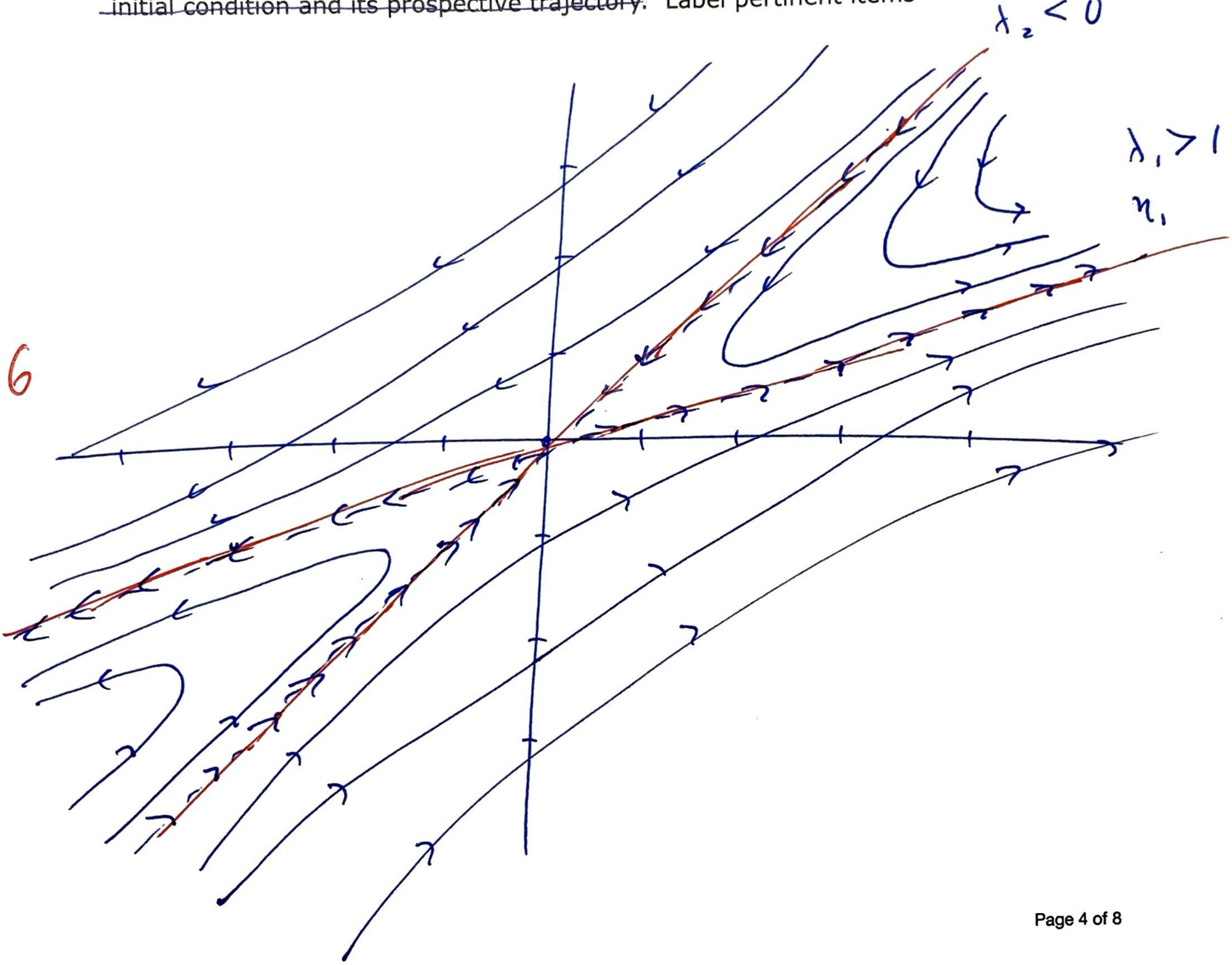

$\boxed{5.}$ (B.2: 10 pts) Convert the ODE $2y''' - 4y'' + 5y' - 4y = 0$ into a system of first order equations (matrix/vector form). { let $x_1=y$, $x_2=y'$, $x_3=y''$}. Create an initial condition vector also if $y(0) = 1$, $y'(0) = -2$, $y''(0) = -3$

3

$$x_1 = y$$
$$x_2 = y'$$
$$x_3 = y''$$

$$\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = (+4y'' - 5y' + 4y)/2 \\ \quad = 2x_3 - \frac{5}{2}x_2 + 2x_1 \end{cases}$$

5

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} x_2 \\ x_3 \\ 2x_1 - \frac{5}{2}x_2 + 2x_3 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5/2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$$

I.C.

2

$$\boxed{\vec{x}(0) = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}}$$

6. (EVP: 10 pts) Find the eigenvalues and eigenvectors of the given matrix:

$$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$$

(a) Find and state the characteristic equation (show all work):

2

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix}$$

$$= (1-\lambda)(5-\lambda) - 12$$

$$= (\lambda-1)(\lambda-5) - 12$$

$$= \lambda^2 - 6\lambda + 5 - 12$$

$$= \boxed{\lambda^2 - 6\lambda - 7}$$

$$= (\lambda+1)(\lambda-7)$$

$$\boxed{\lambda = -1, 7}$$

(b) ***The roots of the characteristic equation are -1 and 7,*** use these eigenvalues to find their corresponding eigenvectors. Show all steps. Use the η symbol for the vectors and their components. Add sheets on blank white paper if needed after this page.

$\boxed{\lambda_1 = -1}$

$$\left(\begin{array}{cc|c} 1-(-1) & 3 & 0 \\ 4 & 5-(-1) & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 6 & 0 \end{array}\right) *-2 \; \downarrow +$$

4

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad 2\eta_1 + 3\eta_2 = 0, \; \eta_1 = -\tfrac{3}{2}\eta_2$$

$$\vec{\eta}_1 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\tfrac{3}{2}\eta_2 \\ \eta_2 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} \frac{\eta_2}{2} \text{ or } \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\boxed{\lambda_2 = 7}$

$$\left(\begin{array}{cc|c} 1-7 & 3 & 0 \\ 4 & 5-7 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -6 & 3 & 0 \\ 4 & -2 & 0 \end{array}\right) \div 2 \quad \left(\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right) *3$$

4

$$\to \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & -1 & 0 \end{array}\right) \to 2\eta_1 - \eta_2 = 0 \quad \eta_2 = 2\eta_1$$

$$\vec{\eta}_2 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \eta_1 \quad \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$