 Name KEY

A $\frac{30}{30}$ & $\frac{30}{30}$ B

Following the HW guidelines, <u>show or explain</u> ALL work for full credit. All problems are 10 pts each unless otherwise noted. *<u>This test is closed text, closed notes, closed internet sites and closed personal help.</u>* The LAST PAGES contain useful information. **Use the notation, symbols and methods that we use in class or see in the text.**

1. [1.2 Existence] (10 pts) Refer to the Theorems in the Last Pages of this test to answer the following:

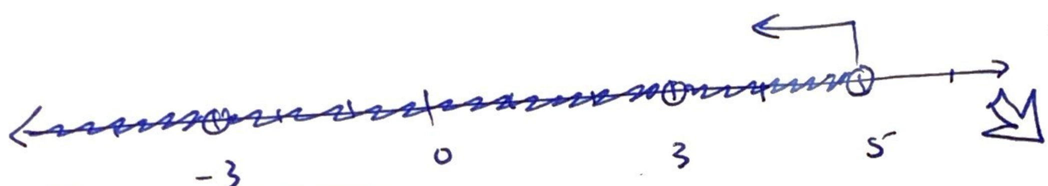
a) Without solving, <u>list</u> the intervals (regions) of validity for the **linear** ODE { Use a number line first }

$$(t^2 - 9)y' + 2y = \sqrt{5-t}$$

Form: $y' + \underbrace{\left(\frac{2}{(t+3)(t-3)}\right)}_{P(t)} y = \underbrace{\left(\frac{\sqrt{5-t}}{(t+3)(t-3)}\right)}_{Q(t)}$

7 $P(t)$ discontinuous @ $t = \pm 3$, $Q(t)$ the same but also is not defined for $t > 5$

so $t \leq 5$



$\boxed{\text{Intervals: } (-\infty, -3) \cup (-3, 3) \cup (3, 5)}$

b) What interval would the initial condition y(2) = -7 belong to?

3 $\boxed{(-3, 3) \text{ since this interval contains } t = 2}$

2. [4.3a Real ≠ roots] (10 pts) Solve ay″ + by′ + c y = 0 where $y(t_0) = 0$ and $y'(t_0) = -7$ where a, b, c = 1, 11, 24 respectively and $t_0 = 0$

$y'' + 11y' + 24y = 0 \quad \begin{cases} y(0) = 0 \\ y'(0) = -7 \end{cases}$

2 (i) List the characteristic polynomial, in r : $r^2 + 11r + 24 = 0$

(ii) Factor or use the quadratic formula to get the roots (show work below):

$(r+8)(r+3) = 0$

24: 2, 12; 6, 4; 8, 3

2 $r_1 = -8$ $r_2 = -3$

(iii) State the general solution form:

2 $y_{gen} = c_1 e^{-8t} + c_2 e^{-3t}$

(iv) Apply the Initial Conditions and state the specific solution:

$t_0 = 0$

$y' = -8c_1 e^{-8t} - 3c_2 e^{-3t}$

$y(0) = c_1 \cdot 1 + c_2 \cdot 1$

$\boxed{0 = c_1 + c_2}$

$y'(0) = -8c_1 \cdot 1 - 3c_2 \cdot 1$

$-7 = -8c_1 - 3c_2$

4 $\boxed{7 = 8c_1 + 3c_2}$

$7 = 8c_1 + 3(-c_1) \rightarrow 7 = 5c_1 \rightarrow \boxed{c_1 = 7/5}$

so $\boxed{c_2 = -7/5}$

$$\boxed{y(t) = \frac{7}{5}e^{-8t} - \frac{7}{5}e^{-3t}}$$

$\boxed{3.}$ [4.3c Complex Roots] (10 pts) Solve ay" + by' + c y = 0 where $y(t_0) = 1$ and $y'(t_0) = 0$ where a, b, c = 1, 2, 5 respectively and $t_0 = 0$

$$y''+2y'+5y=0 \qquad y(0)=1,\ y'(0)=0$$

2 (i) List the characteristic polynomial, in r : $\boxed{r^2+2r+5=0}$

(ii) Factor or use the quadratic formula to get the roots (show work below):

$$r = \frac{-(2) \pm \sqrt{2^2-4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm \frac{\sqrt{-16}}{2} = \boxed{-1 \pm 2i}$$

2

$r_1 =$ $-1+2i$ $r_2 =$ $-1-2i$

(iii) State the general solution form:

2 $y_{gen} = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

(iv) Apply the Initial Conditions and state the specific solution:

$$y(0) = c_1 \cdot 1 \cdot 1 + c_2 \cdot 1 \cdot 0$$

$$\boxed{1 = c_1}$$

$$y_g(t) = e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$y' = -e^{-t}\cos(2t) - 2e^{-t}\sin(2t)$$

4

$$- c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t)$$

$$y'(0) = -1\cdot 1 - 2\cdot 1\cdot 0 - c_2\cdot 1\cdot 0 + 2\cdot c_2\cdot 1\cdot 1$$

$$0 = -1 + 2c_2 \rightarrow \boxed{c_2 = 1/2}$$

Specific:

$$\boxed{y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)}$$

4. [4.2 Reduction] (10 pts) (5pts) Given that $y_1(t) = t$ is a solution of $ty'' - y' + \frac{1}{t}y = 0$ use the reduction of order formula in the last pages to find the general solution of the ODE.

Form : $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0 \Rightarrow P(t) = -\frac{1}{t}, \; Q(t) = \frac{1}{t^2}$

$$y_2(t) = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$= t \int^t \frac{e^{-\int -\frac{1}{x}dx}}{x^2}\, dx, \quad e^{\int \frac{dx}{x}} = e^{\ln|x|} = |x|$$

7

$$= t \int^t \frac{|x|\, dx}{x^2}$$

$$= t \int^t \frac{dx}{x}$$

$$= \underline{t \ln|t|}$$

$$\boxed{y_{gen}(t) = c_1 t + c_2 t \ln|t|}$$

(b) (5 pts) Use the Wronskian to verify linear independence of the functions $y_1 = x$ and $y_2 = 1/x$.

$$W = \begin{vmatrix} t & t\ln|t| \\ 1 & \ln|t| + \frac{t}{t} \end{vmatrix}$$

3

$$= \begin{vmatrix} t & t\ln|t| \\ 1 & 1+\ln|t| \end{vmatrix}$$

$$= (t + t\ln|t|) - t\ln|t|$$

$$\boxed{W = t \neq 0}$$ so y_2 & y_1 are L. Indep.

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}$$

$$= -(x)/x^2 - 1/x$$

$$= -\frac{1}{x} - \frac{1}{x} = \boxed{-\frac{2}{x} \neq 0}$$

Lin Indep.

5. [4.3b Eql Roots] (10 pts) Solve ay″ + by′ + c y = 0 where $y(t_0) = 2$ and $y'(t_0) = \frac{1}{3}$ where a, b, c = 1, -1, $\frac{1}{4}$ respectively and $t_0 = 0$

$y'' - y' + \frac{1}{4}y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{3}$

2 (i) List the characteristic polynomial, in r : $r^2 - r + 1/4 = 0$

(ii) Factor or use the quadratic formula to get the roots (show work below):

$(r - 1/2)^2 = 0$

2 $r_1 =$ $1/2$ $r_2 =$ $1/2$

(iii) State the general solution form:

2 $y_{gen} =$ $c_1 e^{t/2} + c_2 t e^{t/2}$ $= (c_1 + c_2 t)e^{t/2}$

(iv) Apply the Initial Conditions and state the specific solution:

$y(0) = c_1 \cdot 1 + c_2 \cdot 0 \cdot 1$

$\boxed{2 = c_1}$

So $y = 2e^{t/2} + c_2 t e^{t/2}$

$y' = \frac{2}{2}e^{t/2} + c_2 \cdot 1 \cdot e^{t/2} + c_2 \cdot t \cdot \frac{1}{2}e^{t/2}$

2 $y' = (1 + c_2 + \frac{t}{2}c_2)e^{t/2}$

@ $t = 0$ $y'(0) = (1 + c_2 + \frac{0}{2}c_2)e^{0/2}$

$\frac{1}{3} = (1 + c_2) \cdot 1$

2 $\boxed{-\frac{2}{3} = c_2}$

$$\boxed{y_{specific}(t) = 2e^{t/2} - \frac{2}{3}te^{t/2}}$$

6. [4.4 Under Coeff] (10 pts) Superposition: Determine the form of the particular solution of the differential equation by following the steps below. Do not evaluate the coefficients.

$$y'' + 4y = 158t^2 - 3te^{2t} + 7\sin(t) + 25\sin(2t)$$

(a) Solve the homogeneous equation:

$r^2 + 4 = 0 \rightarrow r = \pm 2i$

2 $\boxed{y_c = c_1\cos(2t) + c_2\sin(2t)}$

(b) For each of the RHS terms, write the form of the appropriate particular solution:

2 $\boxed{y_{p1} = At^2 + Bt + C}$

2 $\boxed{y_{p2} = (At + B)e^{2t}}$

2 $\boxed{y_{p3} = A\cos(t) + B\sin(t)}$

2 $\boxed{y_{p4} = t(A\cos(2t) + B\sin(2t))}$

original differed by only a const. so added "t"