Following the HW guidelines, show or explain ALL work for full credit. All problems are 10 pts each unless otherwise noted. *This test is closed text, closed notes, closed internet sites and closed personal help.* The LAST PAGES contain useful information. Use the notation/symbols that we use in class or the text.

1. [1.1 Intro] (10 pts) (a) Consider the differential equation and answer the questions below:

$$\frac{d^2y}{dx^2} - 0.1(1-y^2)\frac{dy}{dx} + 9y = 0$$

(van der Pol's equation, triode vacuum tube)

(i) ODE or PDE (circle one) (ii) Order: 2

(iii) Independent variable: x Dependent variable: y

(iv) Linear or Non-Linear (circle one)

(b) Verify that the solution given below satisfies the ODE shown:

Solution: $x+y+e^{xy}=0$ implicit form of answer so try implicit diff'n

Equation: $(1+xe^{xy})\frac{dy}{dx} + (1+ye^{xy}) = 0$

$$\frac{dx}{dx} + \frac{dy}{dx} + e^{xy}\left(\frac{d(xy)}{dx}\right) = 0$$

$$1 + y' + e^{xy}(y + xy') = 0$$

group y'

$$y' + e^{xy}xy' + 1 + ye^{xy} = 0$$

10

$$\boxed{y'(1+xe^{xy}) + (1+ye^{xy}) = 0 \leftarrow \text{this is the ODE}}$$

$\boxed{2.}$ [1.3 Models] (10 pts) The ODE for free-fall is given with its solution below.
Let m = 100 kg, g = 9.8 m/sec², b = 5 kg/sec, and v(0) = 10 m/sec. Find the terminal velocity, v(t), as t becomes very large. Explain your answer.

$$m\frac{dv}{dt} = mg - bv\,, \qquad v(0) = v_0\,.$$

Soln $\rightarrow$ $v = \dfrac{mg}{b} + \left(v_0 - \dfrac{mg}{b}\right)e^{-bt/m}$

Solution: $v(t) = \dfrac{mg}{b} + \cancel{\left(v_0 - \dfrac{mg}{b}\right)e^{-bt/m}} \rightarrow 0$

- as $t \to \infty$ $\boxed{v(t) = \dfrac{mg}{b}}$ since $e^{-\infty} \to 0$

10

$$v(t) = \frac{(9.8\,m/s^2)(100\,kg)}{5\,kg/sec}$$

$$v(t) = \boxed{196\,m/s}$$

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alt approach not requiring a solution:
- terminal velocity is when v does not change so $\dfrac{dv}{dt} = 0$
- Then the ODE is

$$0 = mg - bv$$

$$v = \frac{mg}{b}$$

3. [2.1 Slope-Fields] (10 pts) Build a direction field plot for the autonomous first order ODE by locating the equilibrium solutions: $y' = (y^2 - 1)(y - 2)^2$ Classify the equilibrium solutions as stable, unstable or quasi-stable (aka semi-stable). Show all steps for full credit.

$(y+1)(y-1)\underbrace{(y-2)^2}_{+}$

y-1: − − − − − − − − − 0 + + + + + +

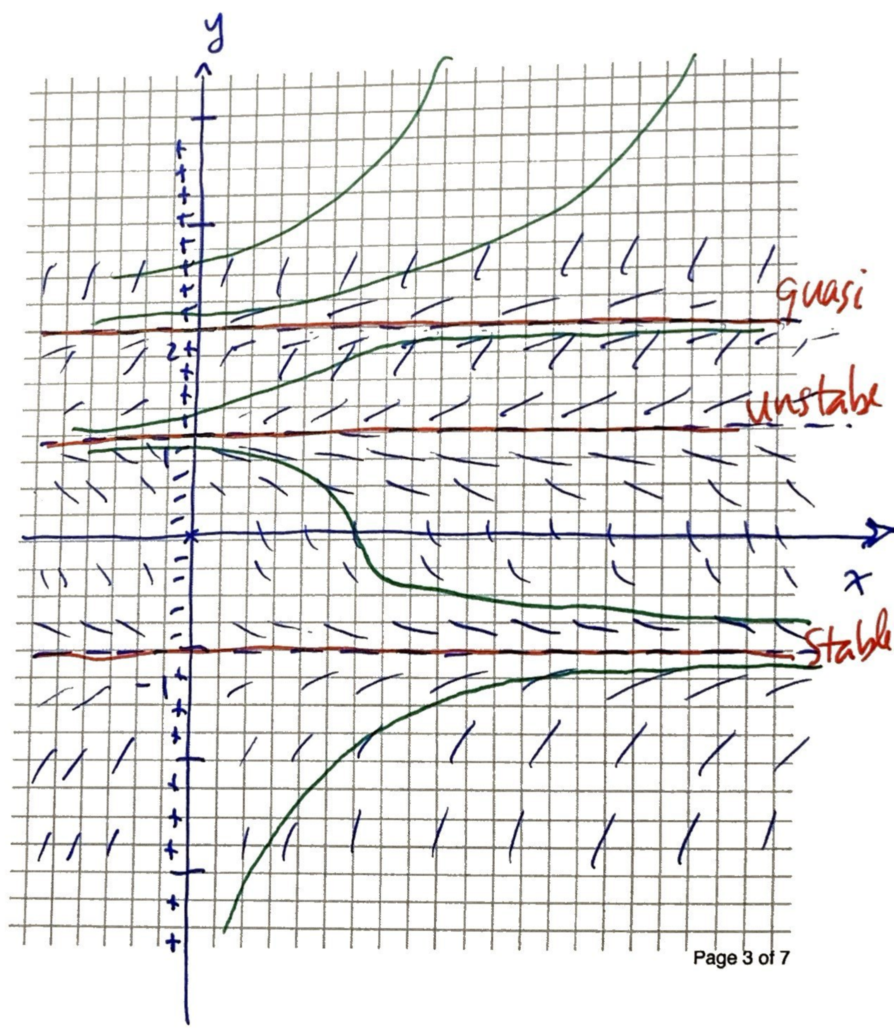
y+1: − − − − − 0 + + + + + + + +

+ + + + + + − − − − + + + + + +

−1 0 1 2 x

$y' = 0$ @ −1, 1, 2

10



$\boxed{4.}$ [2.2 Separable] (10 pts) Use separation of variables to solve: $\dfrac{dy}{dx} = \dfrac{x-5}{y^2}$

$$y^2\,dy = (x-5)\,dx$$

$$\int y^2\,dy = \int (x-5)\,dx$$

10

$$\boxed{\frac{y^3}{3} = \frac{x^2}{2} - 5x + C}$$

5. [2.3 Lin 1st Order] (10 pts) Use an integrating factor to solve the ODE:

$\mu(x) = e^{\int P(x)\,dx}$

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$$

$P(x) = -\frac{2}{x}$

$\int P(x)\,dx$

$= \int \left(-\frac{2}{x}\right) dx$

$= -2 \ln|x|$

$= \ln(x)^{-2}$

so $\mu = e^{\ln(x)^{-2}}$

$\mu = x^{-2}$

$\boxed{\mu(x) = \frac{1}{x^2}}$

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$\frac{1}{x^2}\left(\frac{dy}{dx} - \frac{2}{x}y\right) = \frac{1}{x^2}\left(x^2 \cos x\right)$

$\frac{1}{x^2}y' - \frac{2}{x^3}y = \cos(x)$

$\underbrace{x^{-2}y' - 2x^{-3}y}_{(x^{-2}y)'} = \cos(x)$ 5

$(x^{-2}y)' = \cos(x)$

$\int d(x^{-2}y) = \int \cos(x)\,dx$

$x^{-2}y = \sin(x) + c$

$\boxed{y(x) = x^2 \sin(x) + cx^2}$

6. [2.4 Exact] (10 pts) (a) Verify the given ODE is exact. (b) Solve the exact equation showing the build up of φ(x, y).

$$\underbrace{(2xy - \sec^2 x)}_{M}\, dx + \underbrace{(x^2 + 2y)}_{N}\, dy = 0$$

(a) $\dfrac{\partial M}{\partial y} \overset{?}{=} \dfrac{\partial N}{\partial x}$

2 $\dfrac{\partial (2xy - \sec^2 x)}{\partial y} \overset{?}{=} \dfrac{\partial (x^2 + 2y)}{\partial x}$

$2x \overset{?}{=} 2x$ ✓

(b) use $\boxed{\phi = \int N\, dy}$ {looks easier}

$\phi = \int (x^2 + 2y)\, dy$

4 $\phi = x^2 y + \cancel{2}\dfrac{y^2}{\cancel{2}} + g(x) \rightarrow \boxed{\phi = x^2 y + y^2 + g(x)}$

to determine $g(x)$ use $\boxed{\dfrac{\partial \phi}{\partial x} = M}$

So $\dfrac{\partial (x^2 y + y^2 + g)}{\partial x} = 2xy - \sec^2(x)$

4 $2xy + 0 + g' = 2xy - \sec^2(x)$

$g'(x) = -\sec^2(x)$

$g(x) = -\int \sec^2(x)\, dx$

$g = -\tan(x) + c$

So

$\phi = x^2 y + y^2 - \tan(x)$

$$\boxed{C = x^2 y + y^2 - \tan(x)}$$