Lecture 1st.

B.2 Gaussian Elimination {row ops}

we have learned to solve systems of linear algebraic (not ODE) various ways.

- graph
- substitution
- eliminate (Gaussian)
- Gauss-Jordan
- Cramer's Rule
- Inverse matrix

(*) A <u>system of equations</u> can be written in matrix form

$\boxed{ex}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$\boxed{\mathbb{A}\vec{x} = \vec{b}} \quad \text{if}$$

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

To solve for the $\vec{x}$ vector we can

(a) use the <u>inverse matrix</u>: approach:

$$\mathbb{A}\vec{x} = \vec{b}$$

If $\mathbb{A}^{-1}$ exists $\{\det \mathbb{A} \neq 0\}$, multiply <u>from the left</u> by $\mathbb{A}^{-1}$:

$$\underbrace{\mathbb{A}^{-1}\mathbb{A}}\vec{x} = \mathbb{A}^{-1}\vec{b}$$

But $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$

$$\mathbb{I}\vec{x} = \mathbb{A}^{-1}\vec{b}$$

But $\mathbb{I}\vec{x} = \vec{x}$

$$\boxed{\vec{x} = \mathbb{A}^{-1}\vec{b}}$$ The solution

$\boxed{\text{EX}}$ Solve $\begin{cases} 2x + 3y = 4 \\ 5x - 7y = -1 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 2 & 3 \\ 5 & -7 \end{pmatrix}}_{\mathbb{A}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}_{\vec{b}}$

So $\mathbb{A}^{-1} = \dfrac{\begin{pmatrix} -7 & -3 \\ -5 & 2 \end{pmatrix}}{(2)(-7) - (5)(3)} = \boxed{\frac{-1}{29}\begin{pmatrix} -7 & -3 \\ -5 & 2 \end{pmatrix}}$

then solve the eqn: $\vec{x} = \mathbb{A}^{-1}\vec{b}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{-\frac{1}{29}\begin{pmatrix} -7 & -3 \\ -5 & 2 \end{pmatrix}}_{\mathbb{A}^{-1}} \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}_{\vec{b}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} +\frac{25}{29} \\ \frac{22}{29} \end{pmatrix}$$

$$\boxed{x = \frac{25}{29},\ y = \frac{22}{29}}$$

Elimination Method

$\boxed{\text{Ex}}$ Solve the system

$$\begin{cases}(3x+4y=12)*-2 \\ (2x-3y=-1)*3\end{cases}$$

$$\begin{array}{rl} -6x-8y & =-24 \\ \oplus\ 6x-9y & =-3 \\ \hline 0x-17y & =-27 \end{array}$$

$$\boxed{y=27/17}$$

we back substitute y into either eqn to solve for x

-OR we could continue w/ elimination:

$$\begin{array}{l}(3x+4y=12)*3 \rightarrow \\ (2x-3y=-1)*4 \rightarrow \end{array} \begin{array}{rl} 9x+12y & =36 \\ \oplus\ 8x-12y & =-1 \\ \hline 17x-0y & =35 \end{array}$$

$$\Rightarrow \boxed{x=35/17}$$

ans: $\boxed{(x,y)=\left(\frac{35}{17},\frac{27}{17}\right)}$

augmented matrix

matrix form

$$\begin{aligned} 3x + 4y &= 12 \\ 2x - 3y &= -1 \end{aligned} \rightarrow \begin{pmatrix} 3 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$$

- write in augmented form:

$$\left(\begin{array}{cc|c} 3 & 4 & 12 \\ 2 & -3 & -1 \end{array}\right)$$

augmented form

- we then use row operations to turn the matrix into this form

$$\left(\begin{array}{cc|c} \# & \# & \# \\ 0 & \# & \# \end{array}\right)$$

- answer for one var.
- back substitute to get the other value

(*) Row operations:

1: We can exchange rows w/o penalty

$$R_i \leftrightarrow R_k$$

2: multiply an entire row by a number (or var)

$$cR_i \rightarrow R_k$$

3: multiply an entire row by a number and add to another row replacing it.

$$cR_i + R_k \rightarrow R_k$$

Two by Two augmented matrix:

$$\begin{aligned} 3x + 4y &= 12 \\ 2x - 3y &= -1 \end{aligned} \rightarrow \left(\begin{array}{cc|c} 3 & 4 & 12 \\ 2 & -3 & -1 \end{array}\right) \div 3$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 4/3 & 4 \\ 2 & -3 & -1 \end{array}\right) * -2 \Rightarrow \left(\begin{array}{cc|c} 1 & 4/3 & 4 \\ 0 & -17/3 & -9 \end{array}\right)$$

Convert back into eqn form:

$$1 \cdot x + \frac{4}{3} \cdot y = 4$$
$$0 \cdot x - \frac{17}{3} y = -9 \quad * \left(-\frac{3}{17}\right)$$

$$\Rightarrow \quad x + \frac{4}{3} y = 4$$
$$y = -9\left(\frac{-3}{17}\right)$$
$$\boxed{y = \frac{27}{17}}$$

back substitute

$$x + \frac{4}{\cancel{3}}\left(\frac{\overset{9}{\cancel{27}}}{17}\right) = 4$$

$$x = 4 - \frac{36}{17}$$

$$x = \frac{68 - 36}{17}$$

$$\boxed{x = \frac{32}{17}}$$

Ans: $\boxed{(x, y) = \left(\frac{32}{17}, \frac{27}{17}\right)}$

ex 3x3 augmented matrix

$$\left.\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 \quad + x_3 &= -1 \end{aligned}\right\} \Rightarrow \begin{array}{c} \begin{array}{ccc} & \text{4th} & \text{3rd} \end{array} \\ \left(\begin{array}{ccc|c} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{array}\right) \text{swap} \\ \begin{array}{ccc} \text{1st} & \text{2nd} & \end{array} \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{array}\right) *2 \,;\, *-3 \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{array}\right)$$

$R_2/5,\ R_3/2$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -3 & -4 & -20 \end{array}\right) *3 \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -1 & -2 \end{array}\right) ; *3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -2 \end{array}\right) \begin{array}{l} \\ *-2 \\ \div -1 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

eqn "space"

$$\begin{aligned} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= -1 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 &= 4 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 &= 2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -1 \\ x_2 &= 4 \\ x_3 &= 2 \end{aligned}$$

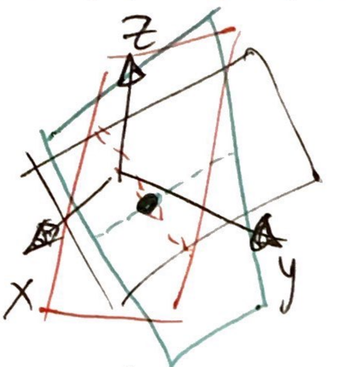
ans:

$$\boxed{(x_1, x_2, x_3) = (-1, 4, 2)}$$

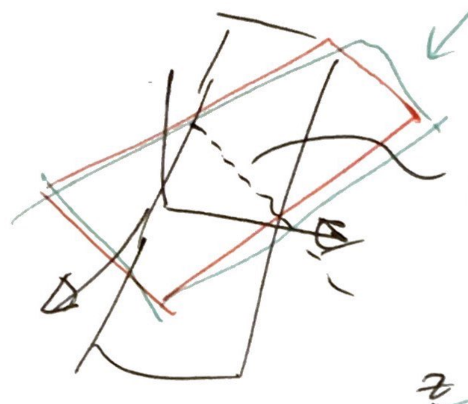
<u>No solution</u>: $ax + by + cz = b$ is a plane in 3-D

a syst of 3 eqns w/ 3 unknowns is three plans.

various outcomes:



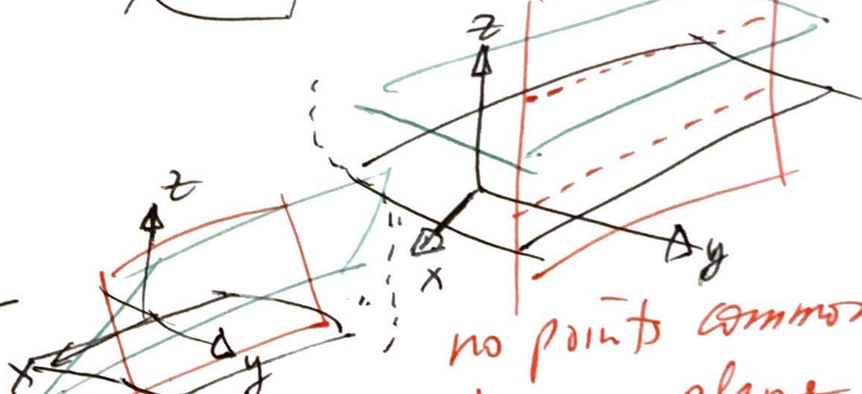
only one point common to all three planes



coplanar planes

common to all 3
∞-many solns

-OR-



no points common to any plane (no-solution)

Q: How do we know there is no soln or ∞-many solutions?

$\boxed{EX}$

$$\begin{cases} x_1 - 2x_2 + 3x_3 = -2 \\ -x_1 + x_2 - 2x_3 = 3 \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{array}\right)$$

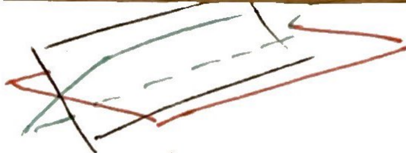
$$\xrightarrow[\text{ops}]{\text{row}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{array}\right) \rightarrow \begin{array}{l} x_1 - 2x_2 + 3x_3 = -2 \\ x_2 - x_3 = -1 \\ \boxed{0x_3 = 8} \end{array}$$

also $\det(A) = 0$

$0 = 8$?

0 solutions

∞-many solutions



$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= -7 \end{aligned}\right\} \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{array}\right)$$

$$\xrightarrow[\text{ops}]{\text{row}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \infty \text{ solution.}$$

$$\xrightarrow[\text{space}]{\text{eqn.}} \begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 - x_3 &= -1 \end{aligned}$$

let $\boxed{x_3 = s}$ assignment

then

$$\begin{aligned} x_1 - 2x_2 + 3s &= -2 \\ x_2 - s &= -1 \to \boxed{x_2 = -1+s} \end{aligned}$$

$$\begin{aligned} x_1 - 2(-1+s) + 3s &= -2 \\ x_1 + 2 - 2s + 3s &= -2 \\ x_1 + s &= -2 - 2 \Rightarrow \boxed{x_1 = -4-s} \end{aligned}$$

ans: $\boxed{(x_1, x_2, x_3) = (-4-s, -1+s, s)}$ ∞-many solutions

BTW: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4-s \\ -1+s \\ s \end{pmatrix}$

OR $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ s \\ s \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} s$

parametric eqn. of a line in 3D

2-parameter solution

$$\left.\begin{aligned} x_1+x_2+x_3&=1\\ 2x_1+2x_2+2x_3&=2\\ 3x_1+3x_2+3x_3&=3\end{aligned}\right\} \to \left(\begin{array}{ccc|c} 1&1&1&1\\ 2&2&2&2\\ 3&3&3&3\end{array}\right) \begin{array}{l} *-2\,;\,*-3 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1&1&1&1\\ 0&0&0&0\\ 0&0&0&0\end{array}\right) \xrightarrow{\text{eqn. space}} x_1+x_2+x_3=1$$

let $\boxed{x_3=s,\quad x_2=t}$ then $x_1+t+s=1$

or $\boxed{x_1=1-t-s}$

ans $\boxed{(x_1,x_2,x_3)=(1-t-s,\ t,\ s)}$

BTW: $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1-t-s\\t\\s\end{pmatrix}$

OR $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}-t\\t\\0\end{pmatrix}+\begin{pmatrix}-s\\0\\s\end{pmatrix}$

$$\boxed{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\0\end{pmatrix}t+\begin{pmatrix}-1\\0\\1\end{pmatrix}s}$$

parametrized plane

$s,t\in\mathbb{R}$

$\vec{x}=f(s,t)$

$\boxed{\text{two params} = \text{surface}}$