Appendix B.1 Matrix Review

Recall in Pre-Calc. systems of equations can be written as a matrix

But first what is a matrix :

A matrix is an array of numbers.

ex: $\mathbb{A} = \begin{pmatrix} 1 & -7 & 0 \\ 4 & 2 & 10 \end{pmatrix}$ this is a 2×3 ← rows ← columns

$\vec{v} = (11, -4, 7)$ this is a 1×3 matrix a.k.a. "vector"

$\vec{u} = \begin{pmatrix} 0 \\ 14 \\ -27 \end{pmatrix}$ this is a 3×1 matrix allso reffered to as a vector.

Addition

* we can add matrices that are off the same dimension:

$$\mathbb{A} + \mathbb{B} = \overbrace{\begin{pmatrix} 1 & 5 \\ 3 & -6 \end{pmatrix}}^{\mathbb{A}} + \overbrace{\begin{pmatrix} 2 & 8 \\ 0 & -4 \end{pmatrix}}^{\mathbb{B}} = \begin{pmatrix} 1+2 & 5+8 \\ 3+0 & -6-4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3 & 13 \\ 3 & -10 \end{pmatrix}}$$

- subtraction is the addition of a negated matrix.
- negation

if $\mathbb{A} = \begin{pmatrix} 18 & 3 & 4 \\ 1 & 7 & -1 \end{pmatrix}$ then $-\mathbb{A} = \begin{pmatrix} -18 & -3 & 4 \\ -1 & 7 & 1 \end{pmatrix}$

$$\mathbb{A} - \mathbb{B} \equiv \mathbb{A} + (-\mathbb{B})$$

$\boxed{\text{ex}}$ let $\mathbb{A} = \begin{pmatrix} 1 & 5 & -7 \\ 0 & -1 & 4 \\ 2 & 3 & 0 \\ 0 & 3 & 7 \end{pmatrix}$ let $\mathbb{B} = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 2 \\ 6 & 6 & -8 \\ 1 & 0 & 4 \end{pmatrix}$

then $\mathbb{A} - \mathbb{B}$

$$= \mathbb{A} + (-\mathbb{B})$$

$$= \begin{pmatrix} 1 & 5 & -7 \\ 0 & -1 & 4 \\ 2 & 3 & 0 \\ 0 & 3 & 7 \end{pmatrix} + \begin{pmatrix} -1 & 2 & -4 \\ -1 & 0 & -2 \\ -6 & -6 & 8 \\ -1 & 0 & -4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 & 7 & -11 \\ -1 & -1 & 2 \\ -4 & -3 & 8 \\ -1 & 3 & 3 \end{pmatrix}}$$

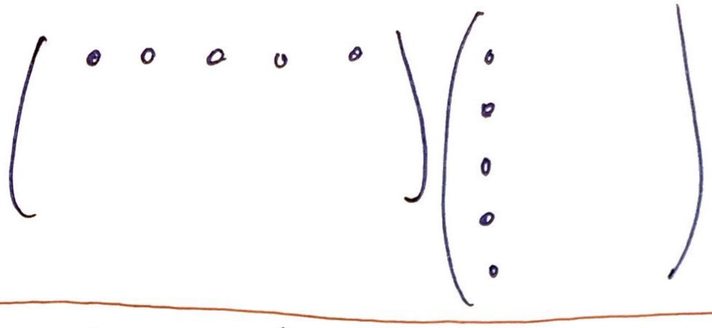
$\boxed{\text{ex}}$

let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ & $\vec{v} = (-4, 0, 8)$

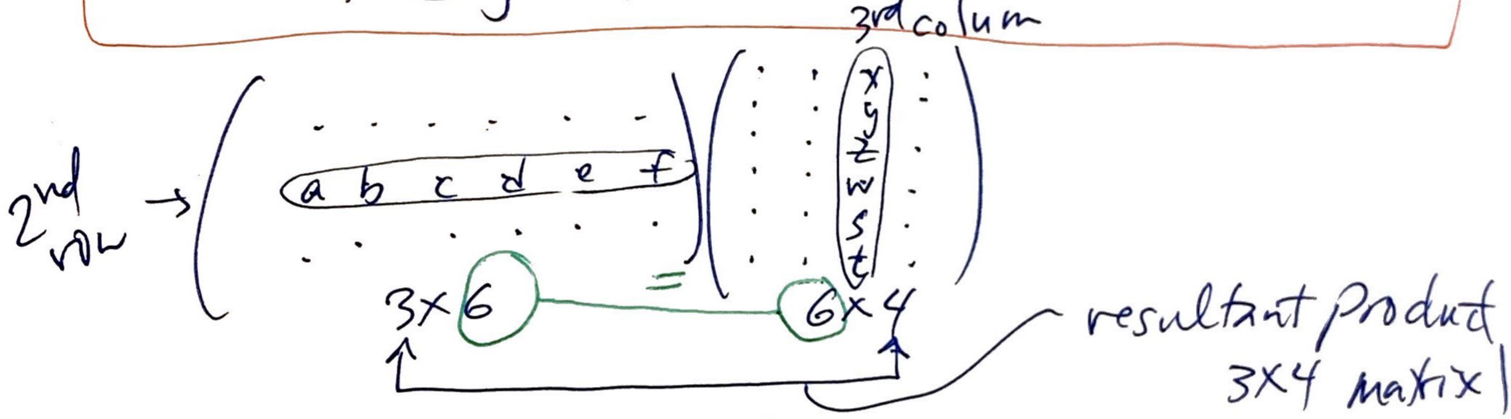
what is $\vec{u} - \vec{v}$?

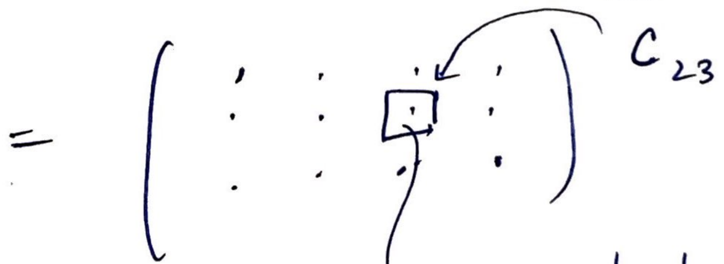
Ans- Incompatable vectors.

multiplication: to <u>multiply matrices</u> the 1st matrix' column count must equal the 2nd matrix' row count.



Def: the ith row, jth col is the product & sum of the ith row of the 1st matrix & the jth column of the 2nd matrix)





$a \cdot x + b \cdot y + c \cdot z + d \cdot w + e \cdot s + f \cdot t$

$\boxed{\text{ex}}$ let $\mathbb{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ let $\mathbb{B} = \begin{pmatrix} 7 & 8 & 13 \\ 9 & 10 & 14 \\ 11 & 12 & 15 \end{pmatrix}$

Find $\mathbb{A}\mathbb{B}$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 13 \\ 9 & 10 & 14 \\ 11 & 12 & 15 \end{pmatrix}$$

$$\underbrace{2\times 3 \quad 3\times 3}_{} \longrightarrow 2\times 3$$

$$= \begin{pmatrix} 1\cdot 7 + 2\cdot 9 + 3\cdot 11 & 1\cdot 8 + 2\cdot 10 + 3\cdot 12 & 1\cdot 13 + 2\cdot 14 + 3\cdot 15 \\ 4\cdot 7 + 5\cdot 9 + 6\cdot 11 & 4\cdot 8 + 5\cdot 10 + 6\cdot 12 & 4\cdot 13 + 5\cdot 14 + 6\cdot 15 \end{pmatrix}$$

$$= \begin{pmatrix} 58 & 64 & 86 \\ 149 & 154 & 210 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 58 & 64 & 86 \\ 149 & 154 & 210 \end{pmatrix}}$$

Not that even if compatibility allows, in general $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$

matrix multiplication does not commute in general.

multiplicative Identity:

There is a matrix, $\mathbb{I}$, when multiplied to any matrix, the resulting matrix remain unchanged.

multiplicative identity matrix

$$\mathbb{A}\,\mathbb{I} = \mathbb{A}$$

- we need $\mathbb{I}$ to be compatible; $\mathbb{I}$ is square.

$\boxed{\text{ex}}$

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a diagonal matrix with 1's in the diagonal, 0's everywhere else.

$\boxed{\text{ex}}$ multiplying from the left.

$$\mathbb{I}\,\mathbb{A} = \mathbb{A}$$

$$\underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}$$

(*) additive identity

$$\mathbb{A} + \boxed{\mathbb{O}} = \mathbb{A} \qquad \mathbb{O} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

* Distributive Law

$A(B+C) = AB + AC$ from the left

$(E+F)G = EG + FG$ from the right

(*) The determinant of a matrix:

The determinant of a matrix results in a scalar.

$$\text{array} \xrightarrow{\det} \text{scalar.}$$

$\boxed{\text{EX}}$ Cofactor expansion: we can pick any row or column and calculate the det by summing up the elements in that row or column once multiplied by the "cofactor" of that position.

$\boxed{\text{EX}}$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

sign grid

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

sign

$$= (-)(4)\det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + (+)5\det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$$

$$+ (-)(-6)\det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

$$= -4\{(-)3\det(8) + (+)9\det(2)\} + 5\{(-)7\det(3) + (+)9\det(1)\}$$
$$+ 6\{(+)1\cdot\det(8) + (-)7\det(2)\} = -4(-24+18 \quad -21+9 + 6(8-14) =$$

- determinants of 2×2 & 3×3 : Shortcuts

  - $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{a\cdot d - c\cdot b}$

    ex: $\begin{vmatrix} 4 & 1 \\ -3 & 6 \end{vmatrix} = 4\cdot 6 - (-3)(1) = 24+3 = \boxed{27}$

  - $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$= \left(\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right) = (aei + bfg + cdh) - (gef + hfa + idb) = \text{value}$$

$\boxed{\text{EX:}}$

$$\left|\begin{array}{ccc|cc} 2 & 4 & -1 & 2 & 4 \\ 0 & 8 & -7 & 0 & 8 \\ -5 & 3 & 6 & -5 & 3 \end{array}\right| = (96+140+0) - (40-42+0) = 236 - (-2) = \boxed{238}$$

verify on matrixcalc.org

- If $\det A = 0$ then we call the matrix $\boxed{\text{"singular"}}$

o for

3.6

* Transpose

if $\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$

then $\mathbb{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$

$\boxed{\text{Ex}}$

$$\begin{pmatrix} 1 & 2 \\ 5 & 7 \\ 0 & -3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 7 & -3 \end{pmatrix}$$

* Inverse of a matrix

If $\mathbb{B}\mathbb{A} = \mathbb{I}$ the $\mathbb{A}$ is the inverse of $\mathbb{B}$

call "$\mathbb{B}^{-1}$"

AND $\mathbb{A}$ is called the inverse of $\mathbb{B}$.

I.E.

$$\mathbb{A}\mathbb{A}^{-1} = \mathbb{I}$$
$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$$

we need $\mathbb{A}$ to be square for it to have an inverse.

: How to calculate the <u>inverse of a matrix</u>.

A: The "armani method" (Gaussian Elimination method)

↳ $[A|X]$ logo

(i) Form the augmented matrix $[A|I]$:

(ii) Perform row operation on $[A|I]$ to obtain the form $[I|B]$

(iii) the RHS is the inverse of A.

$\boxed{\text{EX}}$ Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix}$

multiply top row by 5

(i) $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right]$ $*5$ $+$

(ii) $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 17 & 5 & 1 \end{array}\right] \div 17$

(iii) $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{17} & \frac{1}{17} \end{array}\right]$ $* -3$

$\left[\begin{array}{cc|cc} 1 & 0 & 2/17 & -3/17 \\ 0 & 1 & 5/17 & 1/17 \end{array}\right]$

$$\boxed{A^{-1} = \begin{pmatrix} 2/17 & -3/17 \\ 5/17 & 1/17 \end{pmatrix}}$$

Test. $AA^{-1} = \begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}\frac{1}{17} = \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}\frac{1}{17} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$

- shortcut for 2X2 only: let $\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\det \mathbb{A}}$

- exchange main diagonal elements
- change sign of off diag.
- $\div \det \mathbb{A}$

$\boxed{\text{ex}}$ $\begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix}^{-1} = \dfrac{\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}}{1 \cdot 2 - (-5)(3)}$

$= \dfrac{1}{17}\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} \frac{2}{17} & -3/17 \\ 5/17 & 1/17 \end{pmatrix}}$

No shortcut for 3X3, use "Armani Method"

Invert $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & -2 \\ 4 & 5 & 6 \end{bmatrix}$ {avoid fractions!}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} *-3\ ;\ *-4 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -11 & -3 & 1 & 0 \\ 0 & -3 & -6 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} \\ *3 \\ *-7 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -21 & -33 & -9 & 3 & 0 \\ 0 & 21 & 42 & 28 & 0 & -7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 9 & 19 & 3 & -7 \\ 0 & -3 & -6 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} *3 \\ \\ *3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 9 & 3 & 0 & 0 \\ 0 & 0 & 9 & 19 & 3 & -7 \\ 0 & -9 & -18 & -12 & 0 & 3 \end{array}\right] \begin{array}{l} \\ *-1\ ;\ *2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 3 & 6 & 0 & -16 & -3 & 7 \\ 0 & 0 & 9 & 19 & 3 & -7 \\ 0 & -9 & 0 & 26 & 6 & -11 \end{array}\right] \begin{array}{l} *3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 9 & 18 & 0 & -48 & -9 & 21 \\ 0 & 0 & 9 & 19 & 3 & -7 \\ 0 & -9 & 0 & 26 & 6 & -11 \end{array}\right] \begin{array}{l} \\ \\ *2 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 9 & 0 & 0 & 4 & 3 & 1 \\ 0 & 0 & 9 & 19 & 3 & -7 \\ 0 & -9 & 0 & 26 & 6 & -11 \end{array}\right] \begin{array}{l} \div 9 \\ \div 9 \\ \div -9 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4/9 & 3/9 & -1/9 \\ 0 & 0 & 1 & 19/9 & 3/9 & -7/9 \\ 0 & 1 & 0 & -26/9 & -6/9 & 11/9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 4/9 & 3/9 & -1/9 \\ -26/9 & -6/9 & 11/9 \\ 19/9 & 3/9 & -7/9 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\boxed{A^{-1} = \frac{1}{9}\begin{pmatrix} 4 & 3 & -1 \\ -26 & -6 & 11 \\ 19 & 3 & -7 \end{pmatrix}}$$

matrix calculus

• if $\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\boxed{ex}$ $3\mathbb{A} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$

• if $\mathbb{A} = \begin{pmatrix} t^3 & t^2 \\ \sin(t) & \cos(t) \end{pmatrix}$

then $\mathbb{A}' = \boxed{\begin{pmatrix} 3t^2 & 2t \\ \cos t & -\sin t \end{pmatrix}}$

and $\int \mathbb{A}\,dt = \boxed{\begin{pmatrix} t^4/4 & t^3/3 \\ -\cos(t) & \sin(t) \end{pmatrix}}$