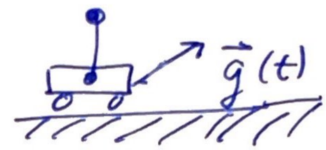
8.3 Non-Homogeneous Systems of 1st Order ODE's

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

driving force



- The methods of chpt 4, Undetermined Coefficient and Variation of Parameters will work for systems.

We will examine first U. Coef., which only applies to the simple driving functions, $e^t, \sin(t), t^2$, etc.

Ex Consider

$$x_1'(t) = x_1(t) + 2x_2(t) + 2t$$
$$x_2'(t) = 3x_1(t) + 2x_2(t) - 4t$$

matrix form:

$$\vec{X} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x} + \underbrace{\begin{pmatrix} 2 \\ -4 \end{pmatrix}}_{\vec{d}} t$$

call this

- complementary solution x_c:

Homog: $\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x} \to \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \to \boxed{\lambda = -1, 4}$

$$\boxed{\lambda_1 = -1 \Rightarrow \vec{\eta}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \lambda_2 = 4 \Rightarrow \vec{\eta}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

$$\boxed{\vec{x}_c(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

- particular solution $\vec{x}_p$ form:

$$\boxed{\vec{x}_p = \vec{a}t + \vec{b}}$$

$\vec{a}$: a_1, a_2 $\quad$ $\vec{b}$: b_1, b_2

plug into non-homogeneous ODE to determine a_1, a_2, b_1, b_2

(Ex cont.) ②

we will need $\vec{x}_p'$: $\frac{d\vec{x}_p}{dt} = \frac{d(\vec{a}t+\vec{b})}{dt}$

$$= \vec{a}\,\cancelto{1}{\frac{dt}{dt}} + \vec{b}\,\cancelto{0}{\frac{d1}{dt}}$$

so $\vec{x}_p' = \vec{a}$

ODE: $\vec{x}_p' = \mathbb{A}\vec{x}_p + \vec{g}$

$$\vec{a} = \mathbb{A}(\vec{a}t+\vec{b}) + \vec{d}t$$

$$(\mathbb{A}\vec{a}+\vec{d})\,t + (\mathbb{A}\vec{b}-\vec{a}) = \vec{0}$$

Solve for coeffient vectors $\vec{a}$ & $\vec{b}$

$$\boxed{\begin{aligned} t^1 &: \mathbb{A}\vec{a}+\vec{d} = \vec{0} \\ t^0 &: \mathbb{A}\vec{b}-\vec{a} = \vec{0} \end{aligned}}$$

→ solve for $\vec{a}$

insert $\vec{a}$

→ solve for $\vec{b}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} 2 \\ -4 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\vec{a}} = -\frac{1}{4}\underbrace{\begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix}}_{\mathbb{A}^{-1}}\underbrace{\begin{pmatrix} -2 \\ +4 \end{pmatrix}}_{\vec{d}} = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{\vec{b}} = -\frac{1}{4}\underbrace{\begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix}}_{\mathbb{A}^{-1}}\underbrace{\begin{pmatrix} 3 \\ -5/2 \end{pmatrix}}_{\vec{a}} = \begin{pmatrix} -11/4 \\ 23/8 \end{pmatrix}$$

So $\boxed{\vec{x}_p = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix} t + \begin{pmatrix} -11/4 \\ 23/8 \end{pmatrix}}$

- General Solution:

$$\boxed{\vec{x}_{gen}(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5/2 \end{pmatrix} t + \begin{pmatrix} -11/4 \\ 23/8 \end{pmatrix}}$$

Now we apply the Initial Conditions

⊛ Variation of Parameters

Let $\mathbb{X} = \left(\vec{x}_1 \middle| \vec{x}_2 \middle| \cdots \middle| \vec{x}_n \right)$ be the fundamental solution matrix of all fundamental solutions of the system $\vec{x}' = \mathbb{A}\vec{x}$

Then $\boxed{\mathbb{X}' = \mathbb{A}\mathbb{X}}$ becomes a differential eqn.

- For $\boxed{\vec{x}' = \mathbb{A}\vec{x} + \vec{g}(t)}$ lets assume the particular solution has the form (per Chpt 4)

$$\boxed{\vec{x}_p = \mathbb{X}\vec{v}} \quad \left\{ = \sum_{i=1}^{n} \vec{x}_i v_i(t), \ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \right.$$

where $\vec{v}(t)$ is a yet to be determined vector function.

- Insert this into the ODE in $\mathbb{X}$:

$$\vec{x}_p' = \mathbb{A}\vec{x}_p + \vec{g}(t)$$

$$(\mathbb{X}\vec{v})' = \mathbb{A}(\mathbb{X}\vec{v}) + \vec{g}(t)$$

product rule

$$\mathbb{X}'\vec{v} + \mathbb{X}\vec{v}' = (\mathbb{A}\mathbb{X})\vec{v} + \vec{g}(t)$$

$$\cancel{\mathbb{X}'\vec{v}} + \mathbb{X}\vec{v}' = \cancel{\mathbb{X}'\vec{v}} + \vec{g}(t)$$

$$\mathbb{X}\vec{v}' = \vec{g}(t) \longrightarrow \boxed{\frac{d\vec{v}(t)}{dt} = \mathbb{X}^{-1}\vec{g}(t)} \quad \text{Solve for } \vec{v}$$

- We can integrate $\vec{v}' = \mathbb{X}^{-1}(t)\,\vec{g}(t)$ to get $\vec{v}(t)$:

$$\boxed{\vec{v}(t) = \int \mathbb{X}^{-1}(t)\,\vec{g}(t)\,dt}$$

- Create $\vec{x}_p(t) = \mathbb{X}\,\vec{v}$

$$\boxed{\vec{x}_p(t) = \mathbb{X}(t)\int \mathbb{X}^{-1}(t)\,\vec{g}(t)\,dt}$$

- General Solution

$$\vec{x}_{gen}(t) = \vec{x}_c(t) + \vec{x}_p$$

let $(y_1 | y_2 | \cdots | y_n)$

$$\boxed{\vec{x}_g(t) = \mathbb{X}(t)\,\vec{c} + \mathbb{X}(t)\int \mathbb{X}^{-1}(t)\,\vec{g}(t)\,dt}$$

$\vec{c} = (c_1, c_2, \cdots, c_n) \leftarrow$ constants.

- parameter form

$$\begin{cases} x_{1g}(t) = c_1 x_{11_c} + c_2 x_{12_c} + \cdots + c_n x_{1n_c} + \boxed{x_{1i}\int \left(\sum y_i g_i\right) dt} \\ x_{2g}(t) = c_1 x_{21_c} + c_2 x_{22_c} + \cdots + c_n x_{2n_c} + x_2 \cdots \\ x_{3g}(t) = \\ \vdots \qquad \vdots \\ x_{ng}(t) = \end{cases}$$

2x2 EX Solve via variation of parameters

$$\vec{x}'(t) = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x} + e^{2t}\begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

(i) Solve the homogeneous $\vec{x}' = A\vec{x}$

(we can go back and see this solution in an earlier 8.2 example)

$$\boxed{\vec{X}_c(t) = c_1 \underbrace{e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\vec{X}_1} + c_2 \underbrace{e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\vec{X}_2}}$$

(ii) Form $\mathbb{X} = (\vec{X}_1 \mid \vec{X}_2)$

$$\boxed{\mathbb{X} = \begin{pmatrix} e^{-t} & -e^{-6t} \\ 4e^{-t} & e^{-6t} \end{pmatrix}}$$

(iii) Invert $\mathbb{X}$: $\mathbb{X}^{-1} = \dfrac{\begin{pmatrix} e^{-6t} & e^{-6t} \\ -4e^{-t} & e^{-t} \end{pmatrix}}{(e^{-t})(e^{-6t}) - (4e^{-t})(-e^{-6t})}$ ↙ $5e^{-7t}$

$$\boxed{\mathbb{X}^{-1} = \begin{pmatrix} \frac{1}{5}e^{t} & \frac{1}{5}e^{t} \\ -\frac{4}{5}e^{6t} & \frac{1}{5}e^{6t} \end{pmatrix}}$$

EX cont.

(iv) Next we need $\mathbb{X}^{-1}\vec{g}$ in the integrand

$$\mathbb{X}^{-1}\vec{g} = \underbrace{\begin{pmatrix} \frac{1}{5}e^{t} & \frac{1}{5}e^{t} \\ -\frac{4}{5}e^{6t} & \frac{1}{5}e^{6t} \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 6e^{2t} \\ -e^{2t} \end{pmatrix}}_{2\times 1} \rightarrow 2\times 1$$

$$\mathbb{X}^{-1}\vec{g} = \begin{pmatrix} \frac{5}{5}e^{3t} \\ -\frac{25}{5}e^{8t} \end{pmatrix}$$

(v) Integrate:

$$\int \mathbb{X}^{-1}\vec{g}\,dt = \begin{pmatrix} \int e^{3t} \\ -5\int e^{8t} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \frac{1}{3}e^{3t} \\ -\frac{5}{8}e^{8t} \end{pmatrix}}$$

(vi) Form $\vec{X}_p = \mathbb{X}\overbrace{\int \mathbb{X}^{-1} g\,dt}$

$$= \begin{pmatrix} e^{-t} & -e^{-6t} \\ 4e^{-t} & e^{-6t} \end{pmatrix}\begin{pmatrix} \frac{1}{3}e^{3t} \\ -\frac{5}{8}e^{8t} \end{pmatrix}$$

$\vec{X}_p$

$$= \begin{pmatrix} \frac{1}{3}e^{-t}e^{3t} + \frac{5}{8}e^{-6t}e^{8t} \\ \frac{4}{3}e^{-t}e^{3t} - \frac{5}{8}e^{-6t}e^{8t} \end{pmatrix} = \begin{pmatrix} \frac{23}{27}e^{2t} \\ \frac{17}{24}e^{2t} \end{pmatrix}$$

(vii) piece things together

$$\vec{x}_g(t) = \Phi\vec{c} + \Phi\int \Phi^{-1}\vec{g}\,dt$$

$$\vec{X}(t) = c_1\vec{X}_{1c} + c_2\vec{X}_{2c} + \frac{e^{2t}}{27}\begin{pmatrix}23\\17\end{pmatrix}$$

$$\boxed{\begin{pmatrix}X_{1gen}\\X_{2gen}\end{pmatrix} = c_1 e^{-t}\begin{pmatrix}1\\4\end{pmatrix} + c_2 e^{-6t}\begin{pmatrix}-1\\1\end{pmatrix} + \frac{e^{2t}}{27}\begin{pmatrix}23\\17\end{pmatrix}}$$

vector form

apply I.C. to get c_1 & c_2

- parametric form

$$\boxed{\begin{aligned} x_{1gen}(t) &= c_1 e^{-t} - c_2 e^{-6t} + \frac{23}{27}e^{2t} \\ x_{2gen}(t) &= c_1 4e^{-t} + c_2 e^{-6t} + \frac{17}{27}e^{2t} \end{aligned}}$$

See you @