$\boxed{8.2\,C}$ 24 JAN

Complex Conjugate Eigenvalues

Ex Solve $\vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

- EVP $\begin{vmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow$ $\lambda_1 = +3\sqrt{3}\,i$, $\lambda_2 = -3\sqrt{3}\,i$

$\boxed{\lambda_1 = 3\sqrt{3}i}$ $(A - \lambda I)\vec{\eta} = 0$

$$\left(\begin{array}{cc|c} 3-3\sqrt{3}i & 9 & 0 \\ -4 & -3-3\sqrt{3}i & 0 \end{array}\right)$$

we know one row will zero out so select the row with the easiest looking eqn.

$$\Rightarrow (3-3\sqrt{3}i)\eta_1 + 9\eta_2 = 0$$

$$\Rightarrow \eta_2 = -\tfrac{1}{3}(1-\sqrt{3}i)\eta_1$$ {pick the component of $\vec{\eta}$ w/o a complex denominator}

- build vector $\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -\frac{1}{3}(1-\sqrt{3}i)\eta_1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}}\frac{\eta_1}{3}$

since $\lambda_2 = \lambda_1^*$ and so $\vec{\eta}_2 = \vec{\eta}_1^*$

$\boxed{\lambda_2 = -3\sqrt{3}i}$ $\boxed{\vec{\eta}_2 = \begin{pmatrix} 3 \\ -1-\sqrt{3}i \end{pmatrix}}$ } we do not need or use these

- gen soln $$\boxed{\vec{x}(t) = c_1 e^{3\sqrt{3}it}\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix} + c_2 e^{-3\sqrt{3}it}\begin{pmatrix} 3 \\ -1-\sqrt{3}i \end{pmatrix}}$$

- For 2nd order ODEs we showed that **both the Real & Imag. parts of $\vec{x}$ yeild solutions** to the ODE individually.
- The same works here in systems...

So we need not persue $\vec{\eta}_2$ and λ_2 case.

Since $x_1(t) = c_1 e^{3\sqrt{3}it}\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$ is the conjugate of $\vec{x}_2(t)$

we need only focus on $\vec{x}_1(t)$.

- Use Euler's eqn: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\vec{x}_1(t) = \left(\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t)\right)\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$$

FOIL

.. multiply through ...

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos(3\sqrt{3}t) + 3i\sin(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - i\sin(3\sqrt{3}t) + \sqrt{3}i\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix}$$

.. separate into Re and Im parts:

$$\vec{x}_1(t) = \underbrace{\begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + i\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ \sqrt{3}\cos(3\sqrt{3}t) - \sin(3\sqrt{3}t) \end{pmatrix}}$$

each part is a Lin. Indep. Soln of the ODE!

- gen soln of $\boxed{\vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\vec{x}}$ is

$$\boxed{\vec{x}(t) = c_1\begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + c_2\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ \sqrt{3}\cos(3\sqrt{3}t) - \sin(3\sqrt{3}t) \end{pmatrix}}$$

no "i"

- apply the initial condition

$$\vec{x}(0) = C_1 \begin{pmatrix} 3\cdot 1 \\ -1-\sqrt{3}\cdot 0 \end{pmatrix} + C_2 \begin{pmatrix} 3\cdot 0 \\ \sqrt{3}\cdot 1 - 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} 3C_1 = 2 \\ -C_1 + \sqrt{3}\,C_2 = -4 \end{array} \right\}$$

$$\xRightarrow{\text{solve}} \boxed{C_1 = 2/3, \quad C_2 = -10/3\sqrt{3}}$$

$\boxed{\vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}}$

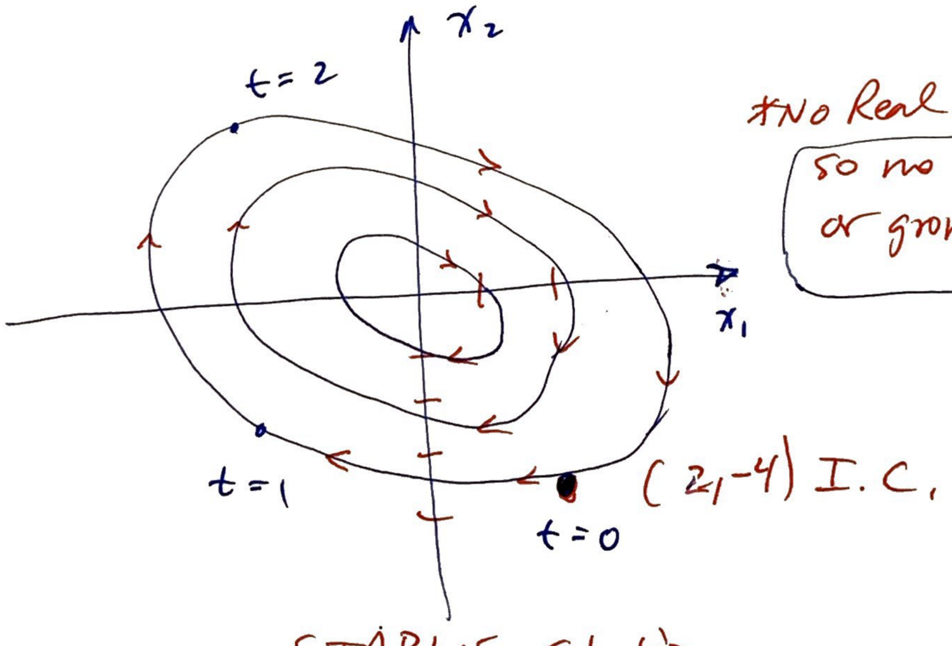
- specific soln to the IVP

$$\vec{x}(t) = \frac{2}{3} \begin{pmatrix} 3\cos 3\sqrt{3}t \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} - \frac{10}{3\sqrt{3}} \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ \sqrt{3}\cos(3\sqrt{3}t) - \sin(3\sqrt{3}t) \end{pmatrix}$$

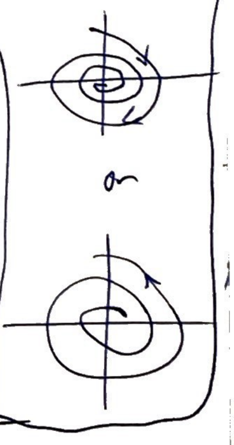
- parametric eqns: $\boxed{x_1(t) = 2\cos(3\sqrt{3}t) - \frac{10}{\sqrt{3}}\sin(3\sqrt{3}t)}$

$$\boxed{x_2(t) = -\frac{2}{3}\cos(3\sqrt{3}t) - \frac{2\sqrt{3}}{3}\sin(3\sqrt{3}t) - \frac{10}{3}\cos(3\sqrt{3}t) + \frac{10}{3\sqrt{3}}\sin(3\sqrt{3}t)}$$

- phase portrait (desmos) • Imag. λ & $\vec{v}$ (no eigenlines)



*No Real part in λ so no decay spiral or growth spiral



STABLE solution

Ex Solve $\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$

- EVP $\quad \det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 68 = 0$$

$$\boxed{\lambda_{1,2} = 2 \pm 8i}$$

$\boxed{\lambda_1 = 2+8i}$ $\quad (A - \lambda I)\vec{n} = \vec{0}$

$$\Rightarrow \left(\begin{array}{cc|c} 3-(2+8i) & -13 & 0 \\ 5 & 1-(2+8i) & 0 \end{array} \right)$$

bottom eqn: $\quad 5n_1 + \underbrace{(1-(2+8i))n_2}_{(-1-8i)n_2} = 0$

$$\Rightarrow n_1 = \frac{1}{5}(1+8i)n_2$$

- build vector:

$$\vec{n}_1 = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}(1+8i) \\ 1 \end{pmatrix} n_2 = \boxed{\begin{pmatrix} 1+8i \\ 5 \end{pmatrix}} \frac{n_2}{5}$$

- Complex soln ($\vec{x}_1$ only) λ_1 $\vec{n}_1$

$$\vec{x}_1 = e^{\boxed{(2+8i)}t} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

{recall we need not persue $\vec{x}_2$}

$$\vec{x}_1 = e^{2t} e^{8it} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$\vec{x}_1 = e^{2t} \left[\cos(8t) + i \sin(8t) \right] \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

- FOIL and separate into Re and Im parts.

$$\vec{x}_1(t) = e^{2t} \begin{pmatrix} \cos 8t - 8\sin 8t \\ 5\cos(8t) \end{pmatrix} + i e^{2t} \begin{pmatrix} 8\cos(8t) + \sin 8t \\ 5\sin 8t \end{pmatrix}$$

- gen soln

$$\boxed{\vec{X}(t) = c_1 e^{2t}\begin{pmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}}$$

- Apply IC: @ $t=0$

$$\begin{pmatrix} 3 \\ -10 \end{pmatrix} = c_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \cdot 1 \cdot \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} c_1 + 8c_2 &= 3 \\ 5c_1 &= -10 \end{aligned}\right\} \quad \boxed{c_1 = -2, \ c_2 = 5/8}$$

- Specific Soln:

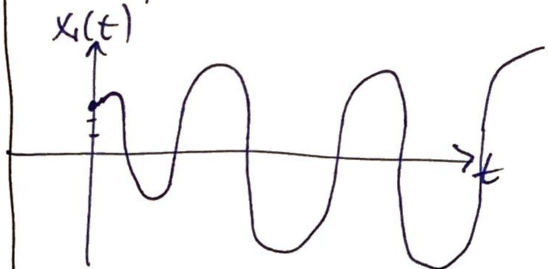
$$\boxed{\vec{X}(t) = -2 e^{2t}\begin{pmatrix} \cos(8t) - 8\sin 8t \\ 5\cos(8t) \end{pmatrix} + \frac{5}{8} e^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}}$$

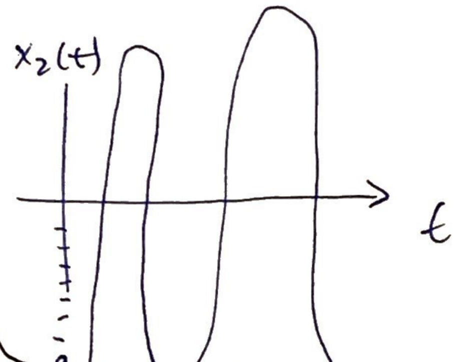
- parametric eqns (simplify above)

$$\boxed{\begin{aligned} x_1(t) &= 3e^{2t}\cos 8t + \frac{131}{8} e^{2t}\sin 8t \\ x_2(t) &= -10\boxed{e^{2t}}\cos 8t + \frac{25}{8}\boxed{e^{2t}}\sin 8t \end{aligned}}$$

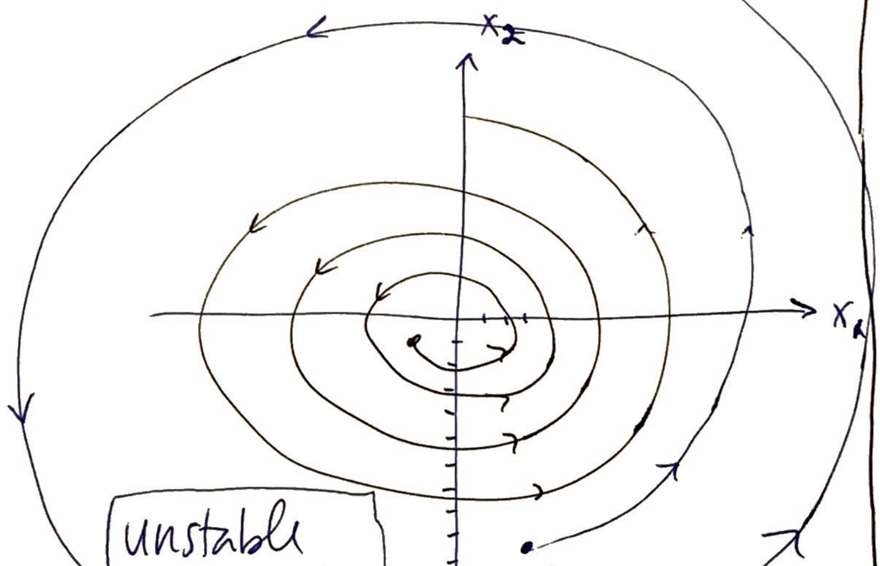
act as an amplitude

- plot $x_1(t)$ & $x_2(t)$





- phase plane



unstable node

b/c $Re(\lambda) > 0$

$a > 0$

Alt. 8.2 c) Complex Conjugate eigenvalues w/Formula

For $\vec{x}' = A\vec{x}$ • Let eigenvalues be $\boxed{\lambda = \alpha \pm i\beta}$

Real   Imag

• Let $\boxed{\vec{\eta} = \vec{a} + i\vec{b}}$ be the eigenvectors.

• Soln: $\underbrace{\vec{w}_1(t) = e^{\lambda_1 t}\vec{\eta}_1 , \vec{w}_2(t) = e^{\lambda_2 t}\vec{\eta}_2}_{\text{Complex solutions}}$

• Use Euler's Identity

$$\boxed{e^{(\alpha \pm i\beta)t} = e^{\alpha t}\cos(\beta t) \pm i e^{\alpha t}\sin(\beta t)}$$

then

$$\vec{w}_{\{1,2\}} = \left(e^{\alpha t}\cos(\beta t) \pm i e^{\alpha t}\sin(\beta t)\right)\left(\vec{a} \pm i\vec{b}\right)$$

• Focus on just $\vec{w}_1$ and FOIL:

Real parts

$$\vec{w}_1 = e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)\left(\vec{a} + i\vec{b}\right)$$

Imaginary Parts.

$$= e^{\alpha t}\left\{\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right\} \leftarrow \text{Real}$$

i^2

$$+ \, i e^{\alpha t}\left\{\cos(\beta t)\vec{b} + \sin(\beta t)\vec{a}\right\} \leftarrow \text{Imag.}$$

• But recall $Re(\vec{w}_1)$ is a solution to $\vec{x}' = A\vec{x}$ as well as the $Im(\vec{w}_1)$ is also a solution of $\vec{x}' = A\vec{x}$

So ...

- Then the general solution is a Lin. Comb. of

$$Re(\vec{w}_1) \text{ \& } Im(\vec{w}_1)$$

{The conjugate of $\vec{w}_1$ is $\vec{w}_2$ and only a sign change is present so we need not persue the results of $\vec{w}_2$}

The Formula

$\lambda = \alpha \pm i\beta$

$\vec{\eta} = \vec{a} + i\vec{b}$

Gen. Solution : $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}(t)$

where $\vec{x}_1(t) = e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b}$

and $\vec{x}_2(t) = e^{\alpha t}\sin(\beta t)\vec{a} + e^{\alpha t}\cos(\beta t)\vec{b}$

EX Solve $\vec{x}' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}\vec{x}$

EVP: $\begin{vmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \underbrace{-2}_{\alpha = -2} \pm \underbrace{i}_{\beta = 1}$

vectors: $\left(\begin{array}{cc|c} -1-(-2+i) & 2 & 0 \\ -1 & -3-(-2+i) & 0 \end{array}\right) \to$

$\to \left(\begin{array}{cc|c} 1-i & 2 & 0 \\ -1 & -1-i & 0 \end{array}\right) \to -\eta_1 - (1+i)\eta_2 = 0$

$\eta_1 = (-1-i)\eta_2$

build $\vec{\eta}_1 = \boxed{\begin{pmatrix} -1-i \\ 1 \end{pmatrix}}\eta_2$, $\vec{\eta} = \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\vec{a}} + i\underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\vec{b}}$

Use the "Formula"

$$\vec{x}_{gen}(t) = c_1\left[e^{-2t}\cos(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} - e^{-2t}\sin(t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + c_2\left[e^{-2t}\sin(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-2t}\cos(t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$$