8.2b Duplicate (real) roots

- We found for duplicate roots in 2^{nd} order ODE's w/ constant coefficients that we ended up just adding a "t" to y_1 to get y_2

- Lets try that here:

$(abc)' = a'bc + ab'c + abc'$

- Assume $\vec{x}_1 = \vec{\eta}_1 e^{\lambda_1 t}$
- Add that "t": $\boxed{\vec{x}_2 = \vec{\eta}_1 \boxed{t} e^{\lambda_1 t}}$
- Diff't: $\boxed{\vec{x}_2{}' = \vec{\eta}_1 \cdot 1 \cdot e^{\lambda_1 t} + \vec{\eta}_1 t \lambda_1 e^{\lambda_1 t}}$
- Insert this into the ODE: $\boxed{\vec{x}' = \mathbb{A}\vec{x}}$

$$\left(\vec{\eta}_1 \cancel{e^{\lambda_1 t}} + \vec{\eta}_1 t \lambda_1 \cancel{e^{\lambda_1 t}}\right) = \mathbb{A}\left(\vec{\eta}_1 t \cancel{e^{\lambda_1 t}}\right) + \vec{0}$$

$\boxed{t^1 : \vec{\eta}_1 \lambda_1 = \mathbb{A}\eta_1}$ this is the original eigenvalue equation $(\mathbb{A} - \lambda\mathbb{I})\vec{\eta}_1 = \vec{0}$ So No New information!

$\boxed{t^0 : \vec{\eta}_1 = \vec{0}}$, but $\vec{\eta}_1$ is NOT $\vec{0}$ ✗

Does not work ... what do we do to fix it? ($\vec{x}_2$, just $\vec{x}_1$ with added "t")

Problem: we had no vector to counter $\vec{\eta}_1$ in the t^0 eqn. So ...

2

* Lets try a different form that accomodates an extra vector call it "$\vec{\rho}$"

$$\boxed{\vec{x}_2(t) = \underbrace{t e^{\lambda_1 t} \vec{\eta}_1}_{t^1} + \underbrace{e^{\lambda_1 t} \vec{\rho}}_{t^0}}$$

- Diff'e :

$$\vec{x}_2{}' = 1 \cdot e^{\lambda_1 t} \vec{\eta}_1 + t \lambda_1 e^{\lambda_1 t} \vec{\eta}_1 + \lambda_1 e^{\lambda_1 t} \vec{\rho}$$

- Insert into ODE :

$$\vec{x}_2{}' = \mathbb{A} \vec{x}_2$$

$$\left(\cancel{e^{\lambda_1 t}} \vec{\eta}_1 + t \lambda_1 \cancel{e^{\lambda_1 t}} \vec{\eta}_1 + \lambda_1 \cancel{e^{\lambda_1 t}} \vec{\rho} \right) = \mathbb{A} \left(t \cancel{e^{\lambda_1 t}} \vec{\eta}_1 + \cancel{e^{\lambda_1 t}} \vec{\rho} \right)$$

- Match Powers:

t^1 : $\lambda_1 \vec{\eta}_1 = \mathbb{A} \vec{\eta}_1 \quad \overset{i.e.}{\Rightarrow} \quad (\mathbb{A} - \lambda \mathbb{I}) \vec{\eta} = \vec{0}$

no new info

t^0 : $\vec{\eta}_1 + \lambda_1 \vec{\rho} = \mathbb{A} \vec{\rho}$ i.e. $\boxed{(\mathbb{A} - \lambda \mathbb{I}) \vec{\rho} = \vec{\eta}_1}$

II

we can now solve for $\vec{\rho}$ then form $\vec{x}_2$

- Strategy : If λ is a duplicate, we will use our eigenvector $\vec{\eta}$ in $(\mathbb{A} - \lambda \mathbb{I}) \vec{\rho} = \vec{\eta}$ to solve for $\vec{\rho}$ then the general solution becomes

$$\boxed{\vec{x}_1(t) = c_1 e^{\lambda_1 t} \vec{\eta}_1 + c_2 \left(t e^{\lambda_1 t} \vec{\eta}_1 + e^{\lambda_1 t} \vec{\rho} \right)}$$

warni : Do NOT use c_3 here, there is no c_3

Repeated eigen values

Solve $\vec{x}'(t) = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x}$ $\quad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

- eigenvalues $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + 25 = 0, \quad (\lambda-5)^2 = 0, \quad \boxed{\lambda_{1,2} = 5}$$

- eigenvector $\boxed{(\mathbb{A} - \mathbb{I}\lambda)\vec{\eta} = \vec{0}}$

$$\left(\begin{array}{cc|c} 7-\lambda & 1 & 0 \\ -4 & 3-\lambda & 0 \end{array}\right) @ \lambda = 5$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right) \rightarrow 2\eta_1 + \eta_2 = 0 \rightarrow \eta_2 = -2\eta_1$$

build $\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}\eta_1$

- need extra helper-vector, $\vec{\rho}$:

$\boxed{(\mathbb{A} - \mathbb{I}\lambda)\vec{\rho} = \vec{\eta}}$ solve for $\vec{\rho}$

$$\Rightarrow \left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right) *2$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

cont...

$\rightarrow$ Top eqn : $2\rho_1 + \rho_2 = 1$

$\Rightarrow$ $\rho_2 = 1 - 2\rho_1$ need more info?

✱ we can use any ρ_1 we desire ✱

so let $\rho_1 = 0$

• build vector: $\vec{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$

$$= \begin{pmatrix} \rho_1 \\ 1-2\rho_1 \end{pmatrix}$$

$$\vec{\rho} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rho_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

But since ρ_1 can be any number, use $\rho_1 = 0$

so $\boxed{\vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

• General Solution

formula: $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{\eta}_1 + c_2 \left(t e^{\lambda_1 t} \eta_1 + e^{\lambda_1 t} \vec{\rho} \right)$

populated...

$$\boxed{\vec{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left[t e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]}$$

gen solution to $\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}$

$\boxed{\text{BTW}}$ w/o picking $\rho_1 = 0$ we could have just inserted $\vec{\rho}$ that we discovered above...

$$\vec{x}(t) = c_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{5t}\left[t\begin{pmatrix}1\\-2\end{pmatrix} + \overbrace{\begin{pmatrix}1\\-2\end{pmatrix}\rho_1 + \begin{pmatrix}0\\1\end{pmatrix}}^{\rho}\right]$$

$$= c_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{5t}\left[t\begin{pmatrix}1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right] + c_2 e^{5t}\rho_1\begin{pmatrix}1\\-2\end{pmatrix}$$

$$= \underbrace{\left[c_1 + \rho_1 c_2\right]}_{\text{pick to be a new } c_1} e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2\left[te^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + e^{5t}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\boxed{\vec{X} = c_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2\left[te^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + e^{5t}\begin{pmatrix}0\\1\end{pmatrix}\right]}$$

Now continue onto the Initial Conditions

$\Downarrow$

- Apply the I.C. @ $t=0$:

$$\begin{pmatrix}2\\-5\end{pmatrix} = c_1 e^{5\cdot 0}\begin{pmatrix}1\\-2\end{pmatrix} + c_2\left[0e^{5\cdot 0}\begin{pmatrix}1\\-2\end{pmatrix} + e^{5\cdot 0}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\boxed{\begin{pmatrix}2\\-5\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}c_1 + \begin{pmatrix}0\\1\end{pmatrix}c_2} \rightarrow \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}2\\-5\end{pmatrix}$$

$\longrightarrow c_1 = 2,\ c_2 = -1$

- specific soln

$$\boxed{\vec{x}(t) = 2e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} - \left[te^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + e^{5t}\begin{pmatrix}0\\1\end{pmatrix}\right]}$$

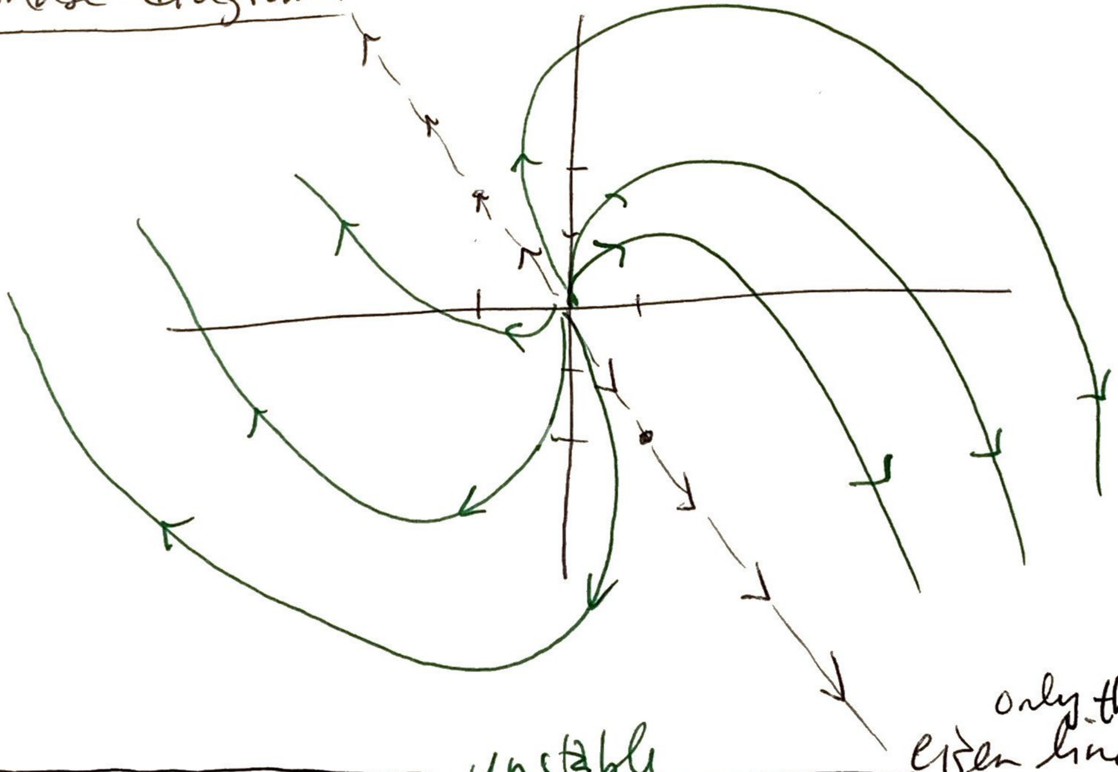
satisfies the ODE and I.C.

- parametric form

$$\boxed{\begin{aligned} x_1(t) &= 2e^{5t} - te^{5t} \\ x_2(t) &= -5e^{5t} + 2te^{5z} \end{aligned}}$$

$\vec{\eta}_1 = \begin{pmatrix}1\\-2\end{pmatrix},\ \lambda = 5 > 0$

- phase-diagram



unstable

only the one eigen line, $\lambda > 0$

EX 3-D w/repeating e.values NOT needing a $\vec{\rho}$ vector

Solve

$$\vec{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \vec{x}$$

• eigenvalue

$$\det(\mathbb{A} - \lambda \mathbb{I}) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)^2(\lambda-5) = 0, \quad \boxed{\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 5}$$

• evectors

$\boxed{\lambda = -1}$

$$\left(\begin{array}{ccc|c} 1-(-1) & -2 & 2 & 0 \\ -2 & 1-(-1) & -2 & 0 \\ 2 & -2 & 1-(-1) & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \quad ; \times -1 \quad ; \div 2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \eta_1 - \eta_2 + \eta_3 = 0$$

$$\Rightarrow \eta_1 = +\eta_2 - \eta_3$$

$$\vec{\eta}_1 = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_2 - \eta_3 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\eta_3 \\ 0 \\ \eta_3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \eta_2 + \boxed{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \eta_3$$

we do not need $\vec{\rho}$ helper vector in this case!

(8)

$\boxed{\lambda_3 = 5}$

$$\left(\begin{array}{ccc|c} 1-5 & -2 & 2 & 0 \\ -2 & 1-5 & -2 & 0 \\ 2 & -2 & 1-5 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \begin{array}{l} \div 2 \\ \div 2; \\ \div 2 \end{array} \quad \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & -2 & -1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} \text{add} ; \times 2 \\ \text{add} \end{array} \quad \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \begin{array}{l} \times -1 ; \\ \div 3 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{add} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

come out of matrixspace:

$$\eta_1 = \eta_3$$
$$\eta_2 = -\eta_3$$
$$\eta_3 = \eta_3 \leftarrow \text{parameter}$$

Build e.vector

$$\vec{\eta}_3 = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_3 \\ -\eta_3 \\ \eta_3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \eta_3$$

Summary

$$\lambda_1 = \lambda_2 = -1 \longrightarrow \vec{\eta}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{\eta}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 5 \rightarrow \vec{\eta}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Ex (cont.)

$\lambda_1 = \lambda_2 = -1 \qquad \lambda_3 = 5$

$$\boxed{\vec{x}(t) = c_1 e^{-t}\underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\vec{n}_1} + c_2 e^{-t}\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{\vec{n}_2} + c_3 e^{5t}\underbrace{\begin{pmatrix}1\\-1\\1\end{pmatrix}}_{\vec{n}_3}}$$

- Apply the I.C.

$$\boxed{\text{let } \vec{x}(0) = \begin{pmatrix}1\\-2\\3\end{pmatrix}}$$

$$\begin{pmatrix}1\\-2\\3\end{pmatrix} = c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$$

Soln

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 1 & 0 & -1 & -2\\ 0 & 0 & 1 & 3\end{array}\right) \begin{array}{l} *-1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & -1 & -2 & -3\\ 0 & 0 & 1 & 3\end{array}\right) \begin{array}{l} \\ \\ *2;\ *-1 \end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & -2\\ 0 & -1 & 0 & 3\\ 0 & 0 & 1 & 3\end{array}\right) \text{add}; \ \div -1$$

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 3\end{array}\right) \rightarrow \boxed{\begin{array}{l} c_1 = 1 \\ c_2 = -3 \\ c_3 = 3 \end{array}}$$

- Sp. Soln

$$\boxed{\vec{x}(t) = e^{-t}\begin{pmatrix}1\\1\\0\end{pmatrix} - 3e^{-t}\begin{pmatrix}1\\0\\1\end{pmatrix} + 3e^{5t}\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

$$\begin{cases} x_1(t) = e^{-t} - 3e^{-t} + 3e^{5t} \\ x_2(t) = e^{-t} - 3e^{5t} \\ x_3(t) = -3e^{-t} + 3e^{5t} \end{cases}$$

ex 3-duplicate evalues needing 2 helper vectors

$$\vec{x}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \vec{x}$$

• EVP $\det \begin{pmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$ $\quad (2-\lambda)(2-\lambda)(2-\lambda) = 0$

$$\boxed{\lambda = 2, 2, 2}$$

$(A - \lambda I)\vec{n} = \vec{0}$

$\boxed{\lambda_1 = 2}$ $\left(\begin{array}{ccc|c} 2-2 & 1 & 6 & 0 \\ 0 & 2-2 & 5 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right) \rightarrow \boxed{\vec{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\vec{p}$: $\left(\begin{array}{ccc|c} 2-2 & 1 & 6 & 1 \\ 0 & 2-2 & 5 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right) \Rightarrow \boxed{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

$(A - \lambda I)\vec{p} = \vec{n}$

$(A - \lambda I)\boxed{\vec{q}} = \vec{p}$ introduce $\vec{q}$ as a 2nd helper vector

$\vec{q}$: $\left(\begin{array}{ccc|c} 2-2 & 1 & 6 & 0 \\ 0 & 2-2 & 5 & 1 \\ 0 & 0 & 2-2 & 0 \end{array}\right) \Rightarrow \boxed{\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}}$

⋮

• gen solution

$$\boxed{\begin{aligned} \vec{x}(t) = c_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{n}_1} e^{2t} + c_2 \left[\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{n}_1} t e^{2t} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{p}} e^{2t} \right] \\ + c_3 \left[\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{n}_1} t^2 e^{2t} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{p}} t e^{2t} + \underbrace{\begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}}_{\vec{q}} e^{2t} \right] \end{aligned}}$$

Now apply I.C. ...

Larger systems

- Consider $\vec{x}\,' = \mathbb{A}\vec{x}$ where $\mathbb{A}$ is an 8X8

assume

$|\mathbb{A} - \lambda \mathbb{I}| = 0$ produces

$$\boxed{\underbrace{\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 2,}\ \lambda_4 = -1+i,\ \lambda_5 = -1-i,\ \lambda_6 = 3i,\ \lambda_7 = -3i,\ \lambda_8 = 144}$$

then the solution forms look like

$$\boxed{\begin{aligned}\vec{x}_{gen} = {} & c_1 e^{2t}\vec{\eta}_1 + c_2\left[e^{2t}t\vec{\eta}_1 + e^{2t}\vec{\rho}\right] + c_3\left[e^{2t}t^2\vec{\eta}_1 + e^{2t}t\vec{\rho} + e^{2t}\vec{q}\right] \\ & + c_4 e^{-t}\cos(t) + c_5 e^{-t}\sin(t) \\ & + c_6\cos(3t) + c_7\sin(3t) \\ & + c_8 e^{144t}\end{aligned}}$$