$\boxed{8.2a}$ Root of the characteristic eqn are unequal

Ex Solve $\left\{\begin{array}{l} x_1' = -5x_1 + x_2 \\ x_2' = 4x_1 - 2x_2 \end{array}\right\}$ w/ $\begin{array}{l} x_1(0) = 1 \\ x_2(0) = 2 \end{array}$

- Matrix Form $\vec{x} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

- e-values $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow = \lambda^2 + 7\lambda + 6$$

$$= (\lambda+1)(\lambda+6) \longrightarrow \boxed{\lambda_1 = -1, \lambda_2 = -6}$$

- evectors

$\boxed{\lambda_1 = -1}$ $\left(\begin{array}{cc|c} -5-(-1) & 1 & 0 \\ 4 & -2-(-1) & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -4 & 1 & 0 \\ 4 & -1 & 0 \end{array}\right)$

Top row: $-4\eta_1 + \eta_2 = 0$ or $\eta_2 = 4\eta_1$

- Build $\vec{\eta}_1 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 4\eta_1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}\eta_1$

$\boxed{\lambda_2 = -6}$ $\left(\begin{array}{cc|c} -5-(6) & 1 & 0 \\ 4 & -2-(-6) & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right)$

Top row: $\eta_1 + \eta_2 = 0 \longrightarrow \eta_1 = -\eta_2$

- Build $\vec{\eta}_2 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}\eta_2$

- gen soln $\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

- IC : $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\4\end{pmatrix} + c_2 e^{-6}\begin{pmatrix}-1\\1\end{pmatrix}$

@ $t=0$ $\begin{pmatrix}1\\2\end{pmatrix} = c_1\cdot 1\cdot\begin{pmatrix}1\\4\end{pmatrix} + c_2\cdot 1\cdot\begin{pmatrix}-1\\1\end{pmatrix}$

$$\Rightarrow \quad \underbrace{\begin{pmatrix}1 & -1\\4 & 1\end{pmatrix}}_{\mathbb{B}}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\mathbb{B}^{-1}: \quad \begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{\begin{pmatrix}1 & 1\\-4 & 1\end{pmatrix}}{5}\begin{pmatrix}1\\2\end{pmatrix} = \underline{\underline{\begin{pmatrix}3/5\\-2/5\end{pmatrix}}}$$

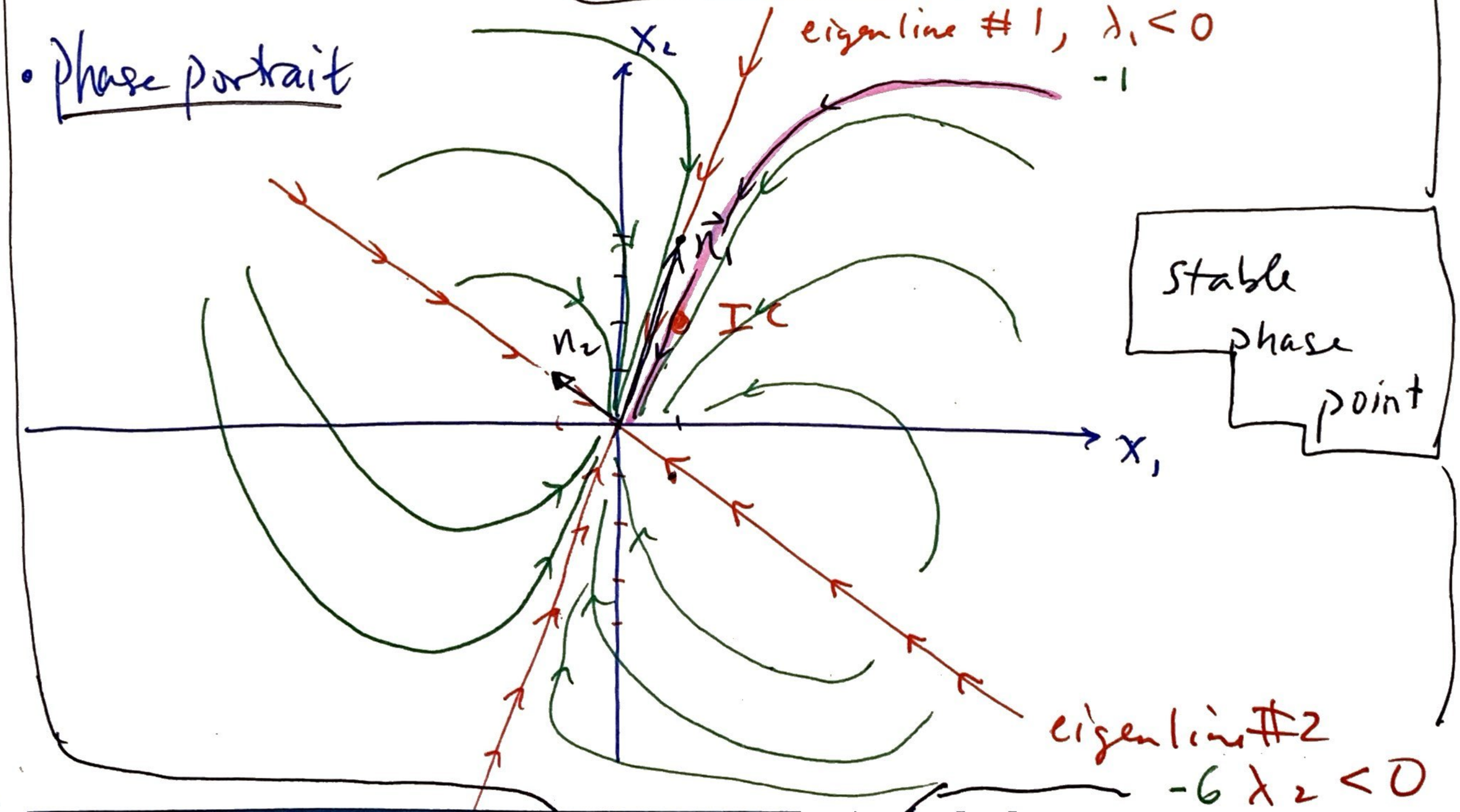
- Specific Soln :

$$\boxed{\vec{x}(t) = \frac{3}{5}e^{-t}\begin{pmatrix}1\\4\end{pmatrix} - \frac{2}{5}e^{-6t}\begin{pmatrix}-1\\1\end{pmatrix}}$$

- Parametric form

$$\boxed{\begin{aligned} x_1(t) &= \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \\ x_2(t) &= \frac{12}{5}e^{-t} - \frac{2}{5}e^{-6t} \end{aligned}}$$

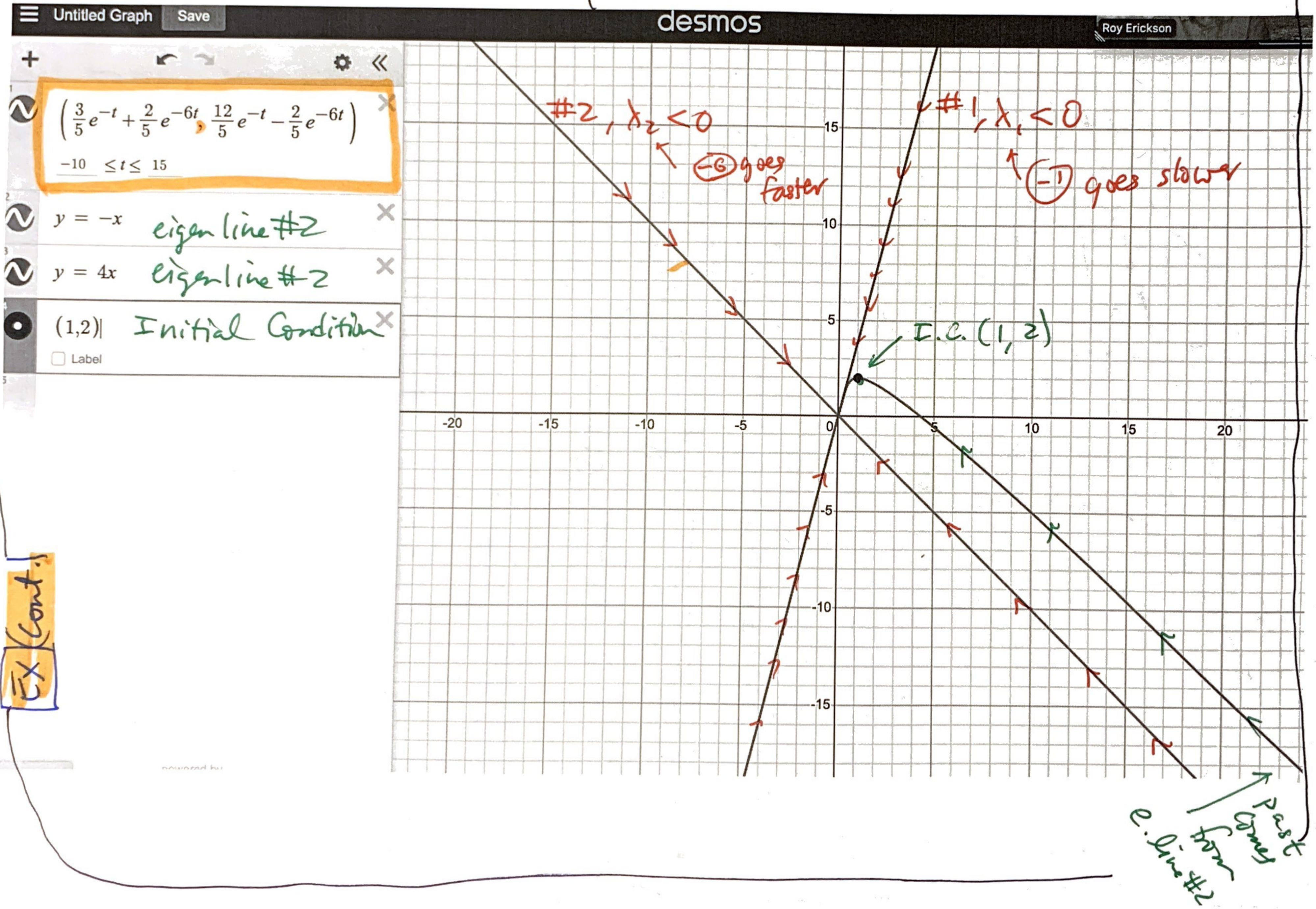
- Phase portrait



Stable phase point

③ parametric Solution of $\left\{\begin{array}{l} x_1' = -5x_1 + x_2 \\ x_2' = 4x_1 - 2x_2 \end{array}\right\}$ w/ $x_1(0) = 1$, $x_2(0) = 2$

Ex (Cont.)



$\left(\frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t}, \frac{12}{5}e^{-t} - \frac{2}{5}e^{-6t}\right)$

$-10 \le t \le 15$

$y = -x$ eigen line #2

$y = 4x$ eigen line #2

(1,2) Initial Condition

☐ Label

(Cont.) Roots of the characteristic Poly are unequal

Ex Solve $2y'' + 5y' - 3y = 0$ w/ $y(0) = -4$, $y'(0) = 9$ via a system of first order ODEs

- let $x_1 = y$ $\quad x_1' = y' = x_2$
  $x_2 = y'$ $\quad x_2' = y'' = \frac{3}{2}x_1 - \frac{5}{2}x_2$ $\Big\}$ $\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 3/2 & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$

- I.C. $\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \boxed{\begin{pmatrix} -4 \\ 9 \end{pmatrix}}$ $\quad A = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}$

- Eigen Problem : $\lambda_1 = -3$, $\vec{n}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\lambda_2 = 1/2$, $\vec{n}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

- General Solution: $\vec{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{\frac{1}{2}t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad t=0$

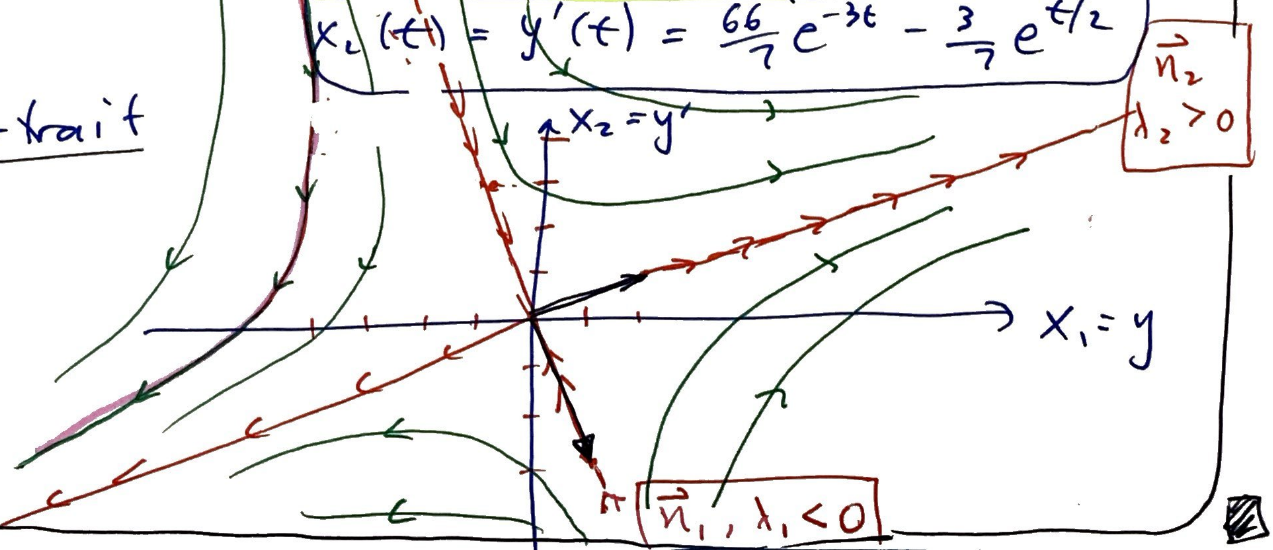
- Apply the I.C. $\begin{pmatrix} 4 \\ -9 \end{pmatrix} = c_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \cdot 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\Longrightarrow \left.\begin{array}{r} c_1 + 2c_2 = -4 \\ -3c_1 + c_2 = 9 \end{array}\right\} \longrightarrow \boxed{\begin{array}{l} c_1 = -22/7 \\ c_2 = -3/7 \end{array}}$$

- Specific Solution: $\boxed{\vec{x}(t) = -\frac{22}{7}e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7}e^{t/2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

- Parametric form $\quad x_1(t) = y(t) = -\frac{22}{7}e^{-3t} - \frac{6}{7}e^{t/2}$
  $x_2(t) = y'(t) = \frac{66}{7}e^{-3t} - \frac{3}{7}e^{t/2}$

- Phase Portrait



FURTHER EXAMPLE

Real and distinct (not equal) e.values

$$\boxed{\vec{x}\,' = \mathbb{A}\vec{x} \text{ has a soln } \vec{x}(t) = c_1 e^{\lambda_1 t}\vec{\eta}_1 + c_2 e^{\lambda_2 t}\vec{\eta}_2}$$

$$\underbrace{\begin{matrix}\lambda_1, \vec{\eta}_1\\ \lambda_2, \vec{\eta}_2\end{matrix}}_{\text{eigen values \& vectors for } \mathbb{A}}$$

Case one: Real and unequal eigenvalues:

EX

$$x_1' = x_1 + 2x_2 \qquad \text{with} \qquad x_1(0) = 0$$
$$x_2' = 3x_1 + 2x_2 \qquad\qquad\qquad x_2(0) = -4$$

(i) need eigenstuff: $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}_{\mathbb{A}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda + 1)(\lambda - 4) = 0 \Rightarrow \boxed{\begin{matrix}\lambda_1 = -1 \\ \lambda_2 = 4\end{matrix}}$$

both real

$$\boxed{\lambda_1 = -1} \qquad (\mathbb{A} - \lambda\mathbb{I})\vec{\eta} = \vec{0}$$

$$\left.\left(\begin{array}{cc|c} 1-\lambda & 2 & 0 \\ 3 & 2-\lambda & 0 \end{array}\right)\right|_{\lambda_1 = -1}$$

$$\Rightarrow \left(\begin{array}{cc|c} 1-(-1) & 2 & 0 \\ 3 & 2-(-1) & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right)$$

$$\Rightarrow 2\eta_1 + 2\eta_2 = 0 \quad \text{or} \quad \boxed{\eta_1 = -\eta_2}$$

Form the eigen vector

$$\vec{\eta}_1 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}\eta_2$$

$\boxed{\lambda_2 = 4}$ $\left.\begin{pmatrix} 1-\lambda & 2 & | & 0 \\ 3 & 2-\lambda & | & 0 \end{pmatrix}\right|_{\lambda_2=4}$

$$\begin{pmatrix} 1-4 & 2 & | & 0 \\ 3 & 2-4 & | & 0 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & | & 0 \\ 3 & -2 & | & 0 \end{pmatrix}$$

$$-3\eta_1 + 2\eta_2 = 0 \qquad \boxed{\eta_1 = \tfrac{2}{3}\eta_2}$$

Form E. vector:

$$\vec{\eta}_2 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2/3\,\eta_2 \\ \eta_2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}\tfrac{1}{3}\eta_2$$

- The gen solution then is

$$\boxed{\vec{X}(t) = c_1 e^{\overset{\lambda_1}{-t}} \overset{\vec{\eta}_1}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} + c_2 e^{\overset{\lambda_2}{4t}} \overset{\vec{\eta}_2}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}}$$

- I.C. $\vec{X}(0) = c_1 e^{-0}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4\cdot 0}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -c_1 + 2c_2 \\ c_1 + 3c_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \text{ or better } \begin{pmatrix} -1 & 2 & | & 0 \\ 1 & 3 & | & -4 \end{pmatrix} +$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & | & 0 \\ 0 & 5 & | & -4 \end{pmatrix} \ast -\tfrac{2}{5} \to \begin{pmatrix} -1 & 0 & | & 8/5 \\ 0 & 5 & | & -4 \end{pmatrix}$$

$$\boxed{c_1 = -8/5 \,, \quad c_2 = -4/5}$$

- Solution:

$$\boxed{\vec{X}(t) = -\tfrac{8}{5}e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \tfrac{4}{5}e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

Break out of vector notation:

$$\left\{ \begin{aligned} x_1(t) &= \tfrac{8}{5}e^{-t} - \tfrac{8}{5}e^{4t} \\ x_2(t) &= -\tfrac{8}{5}e^{-t} - \tfrac{12}{5}e^{4t} \end{aligned} \right.$$

These are parametric curves

- Our last example was the ODE system

$$\text{ODE}\begin{cases}\boxed{\begin{aligned}\frac{dx_1}{dt} &= x_1+2x_2\\ \frac{dx_2}{dt} &= 3x_1+2x_2\end{aligned}}\end{cases}$$

$$\Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}' = \begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \rightarrow \boxed{\begin{aligned}\lambda_1 &= -1,\ \vec{\eta}_1 = \begin{pmatrix}-1\\1\end{pmatrix}\\ \lambda_2 &= 4,\ \vec{\eta}_2 = \begin{pmatrix}2\\3\end{pmatrix}\end{aligned}}$$

$$\Rightarrow \vec{x}(t) = C_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + C_2 e^{4t}\begin{pmatrix}2\\3\end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned}x_1(t) &= -C_1 e^{-t} + C_2 2e^{4t}\\ x_2(t) &= C_1 e^{-t} + C_2 3e^{4t}\end{aligned}}$$

- I.C.: $\vec{x}(0) = \begin{pmatrix}0\\-4\end{pmatrix} \Rightarrow \boxed{\begin{aligned}x_1(0) &= 0\\ x_2(0) &= -4\end{aligned}} \Rightarrow \begin{aligned}C_1 &= -8/5\\ C_2 &= -4/5\end{aligned}$

$$\Rightarrow \vec{x}(t) = -\frac{8}{5}e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} - \frac{4}{5}e^{4t}\begin{pmatrix}2\\3\end{pmatrix}$$

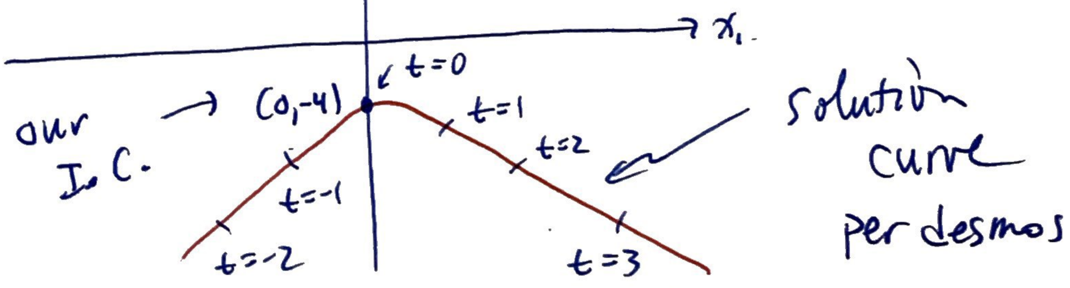
$$\boxed{\begin{aligned}x_1(t) &= \left(\frac{8}{5}e^{-t} - \frac{8}{5}e^{4t}\right)\\ x_2(t) &= \left(-\frac{8}{5}e^{-t} - \frac{12}{5}e^{4t}\right)\end{aligned}}$$

Desmos Entry Window

$$\boxed{((\quad),(\quad))}$$

$$-4 < t < 4$$

- $\boxed{\text{desmos}}$ to graph
{click on ≡ and select parametric example}



Q: How do the eigenvectors manifest themselves on the x_1 vs. x_2 parametric plot we just looked at on Desmos?

- First some Observations {a pause}

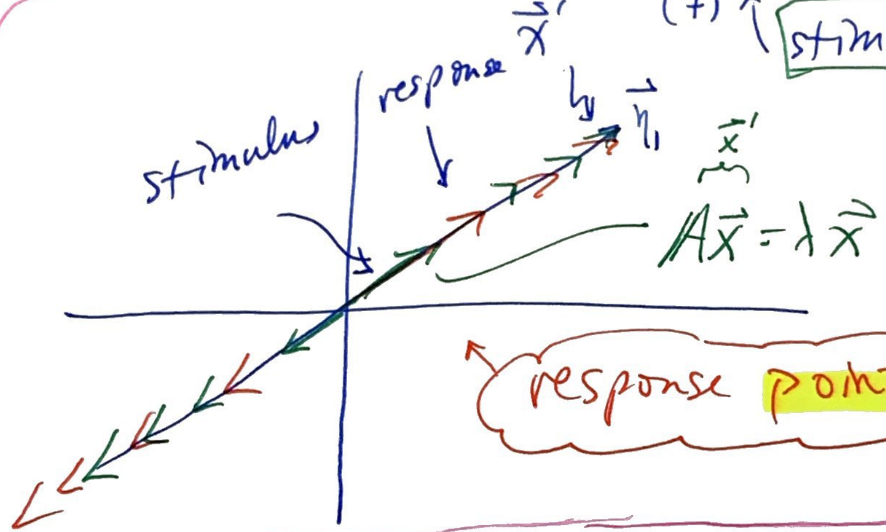
Recall $\underset{\text{input}}{A\vec{x}} = \underset{\text{output}}{\lambda\vec{x}}$ but $\vec{x}' = Ax$ so $\boxed{\vec{x}' = \lambda\vec{x}}$

slope is of the e. vect. is ∥ to the eigenvector - SO - trajectories on the e. vector stay on the eigen.

- If $\boxed{\lambda > 0}$ then $\underset{(+)}{\lambda}\vec{x} = \vec{x}'$

response (slope)

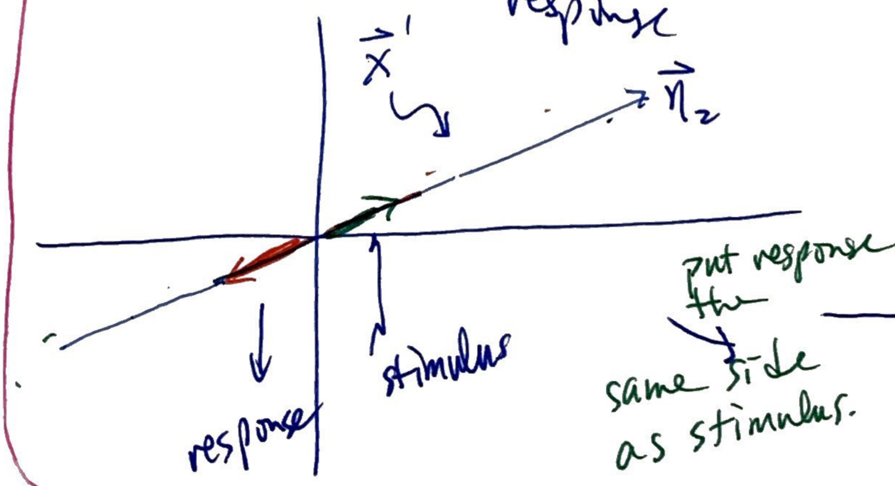
stimulus

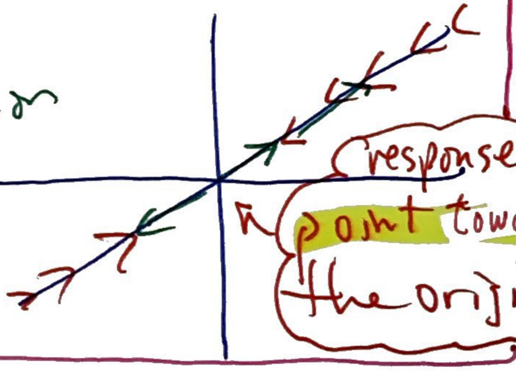


response point away from the origin

- If $\boxed{\lambda < 0}$ — $\boxed{\vec{x}' = \lambda\vec{x}}$

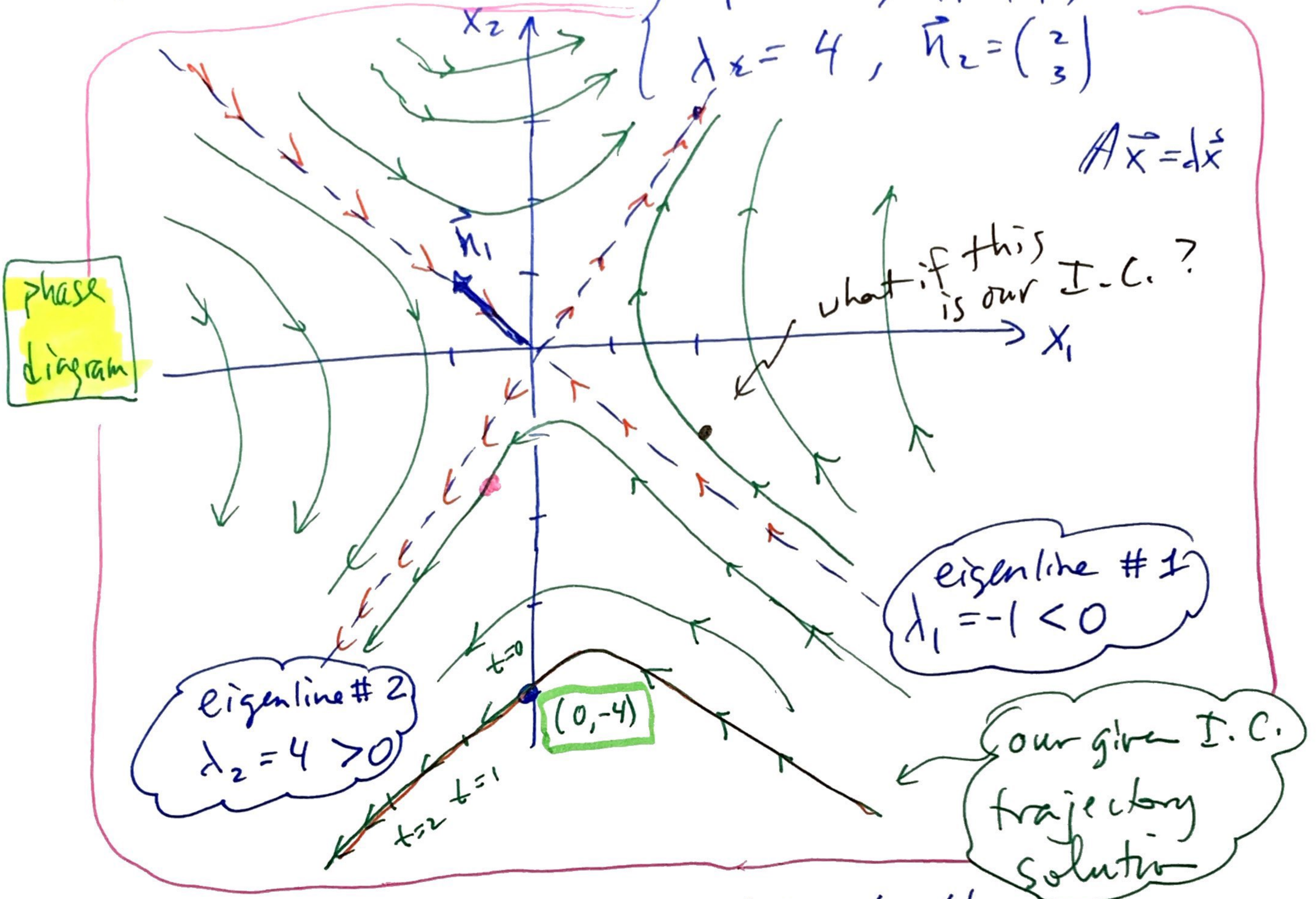
response, stimulus } are in opposite directions





responses point towards the origin

- Back to our ODE: $\begin{cases} \lambda_1 = -1, & \vec{\eta}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \lambda_2 = 4, & \vec{\eta}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{cases}$



- The eigenlines are asymptotes to the trajectory soln when we have real e.values.

- For real, unequal e.values our solution are families of hyperbolas.