8.1 Solving 1st order ODE systems

- We will need to recast systems of ODE's in terms of matrices

Ex

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1 + 2x_2$$
$$\longrightarrow \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

- we need to be able to recast higher order ODEs in 1st order ODE systems

Ex Convert $y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2$

with $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$, $y'''(0) = 4$

(i) chart

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y''' \end{aligned} \xrightarrow{\text{(ii) diff't}} \begin{aligned} x_1' &= y' \\ x_2' &= y'' \\ x_3' &= y''' \\ x_4' &= y^{(4)} \end{aligned}$$

(iii) Sub the RHS to replace y's with x's.

$$x_1' = y' = x_2$$
$$x_2' = y'' = x_3$$
$$x_3' = y''' = x_4$$
$$x_4' = y^{(4)} = t^2 - 8y + \sin(t)\,y' - 3y''$$
$$= t^2 - 8x_1 + \sin(t)\,x_2 - 3x_3$$

use the chart

(iv) Convert to matrix

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}}$$

I.C.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\vec{x}\,' = A(t)\,\vec{x} + \vec{g}\,, \quad \vec{x}(0) = \vec{x}_0$$

⊛ Solving 1st order Linear Systems of ODEs

First we consider an uncoupled system, ie, the $\mathbb{A}$ matrix is diagonal

Ex

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1' = x_1$$
$$x_2' = -4x_2$$
$$x_3' = 6x_3$$

No cross terms!

From Chpt 2 we know via separation of vars

$$x_1(t) = Ce^{t}$$
$$x_2(t) = De^{-4t}$$
$$x_3(t) = Ee^{6t}$$

- In linear algebra there is a matrix $\mathbb{P}$ that is composed of eigenvector of $\mathbb{A}$ such that

$$\mathbb{D} = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}$$

which converts $\mathbb{A}$ into a diagonal matrix whoes entries are the eigenvalues of $\mathbb{A}$

So $\mathbb{P} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\eta}_1 & \vec{\eta}_2 & \cdots & \eta_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$ & $\mathbb{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$

...

- When you work project 3 you will learn that your ODEs, $\boxed{\vec{x}' = \mathbb{A}\vec{x}}$, can be altered by the u-sub $\boxed{\vec{x} = \mathbb{P}\vec{u}} \Rightarrow \vec{x}' = \mathbb{P}\vec{u}'$

Then inserting these into $\vec{x}' = \mathbb{A}\vec{x}$ yeilds

$$\Rightarrow \quad \mathbb{P}\vec{u}' = \mathbb{A}\mathbb{P}\vec{u}$$

- Assuming $\mathbb{P}$ is invertible, $\mathbb{P}^{-1}$ exists, we can multiply the eqn by $\mathbb{P}^{-1}$ from the Left

$$\underbrace{\mathbb{P}^{-1}(\mathbb{P}\vec{u}')}_{} = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}\vec{u}$$

$$\mathbb{I}\vec{u}' = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}\vec{u}$$

$$\boxed{\vec{u}' = \mathbb{P}^{-1}\mathbb{A}\cdot\mathbb{P}\vec{u}}$$

- If we pick $\mathbb{P}$ to be loaded with e.vectors then we know $\mathbb{P}^{-1}\mathbb{A}\mathbb{P}$ is $\mathbb{D}$

I.E.

$$\boxed{\vec{u}' = \mathbb{D}\vec{u}} \xrightarrow[\text{out}]{\text{break}} \begin{cases} u_1' = \lambda_1 u_1 \\ u_2' = \lambda_2 u_2 \\ \vdots \qquad \vdots \\ u_n' = \lambda_n u_n \end{cases}$$

- We can solve this

$$\begin{cases} u_1(t) = c_1 e^{\lambda_1 t} \\ u_2(t) = c_2 e^{\lambda_2 t} \\ \quad\vdots \\ u_n(t) = c_n e^{\lambda_n t} \end{cases}$$

In matrix (vector) form we have

$$\vec{u} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

So $\vec{x} = \mathbb{P}\vec{u}$ becomes

$$\vec{x} = \left(\vec{\eta}_1 \middle| \vec{\eta}_2 \middle| \cdots \middle| \vec{\eta}_n \right) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

general solution

$$\Rightarrow \boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{\eta}_1 + c_2 e^{\lambda_2 t}\vec{\eta}_2 + \cdots + c_n e^{\lambda_n t}\vec{\eta}_n}$$

To solve $\vec{x}' = \mathbb{A}\vec{x}$ we do the following:

(i) Find the eigenvalues & eigenvectors of $\mathbb{A}$

(ii) Form the general solution: $\boxed{\vec{x} = \sum c_i e^{\lambda_i t}\vec{\eta}_i}$

(iii) Apply the I.C. $\vec{x}(0)$ to determine c_i's thus the specif sol

$\boxed{EX}$ We found that for $\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$
the e.values were $\lambda_1 = -1$, $\lambda_2 = 4$ and the corresponding e.vectors were $\vec{\eta}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ & $\vec{\eta}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Using this knowledge, solve the IVP

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned} \quad w/ \quad \begin{aligned} x_1(0) &= 6 \\ x_2(0) &= 7 \end{aligned}$$

(i) matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ w/ $\vec{x}(0) = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

(ii) general solution

$$\boxed{\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

(iii) Apply the I.C.

$$\vec{x}(0) = c_1 e^{-0} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4 \cdot 0} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 6 \\ 7 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

2x2 systems

$$\begin{aligned} 6 &= c_1 \cdot 1 + c_2 \cdot 2 \\ 7 &= c_1 \cdot (-1) + c_2 \cdot 3 \end{aligned} \longrightarrow \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}}^{\mathbb{B}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- Lets use inverse matrices!

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underbrace{\frac{\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}}{1 \cdot 3 - (-1)(2)}}_{\mathbb{B}^{-1}} \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{\begin{pmatrix} 3 \cdot 6 + -2 \cdot 7 \\ 1 \cdot 6 + 1 \cdot 7 \end{pmatrix}}{5} = \begin{pmatrix} 4/5 \\ 13/5 \end{pmatrix} \Longrightarrow$$

the specific solution is then

$$\boxed{\vec{x}(t) = \frac{4}{5}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{13}{5}e^{4t}\begin{pmatrix}2\\3\end{pmatrix}}$$

$\boxed{\text{Test}}$ this Solution: insert into the system $\vec{x}' = \mathbb{A}\vec{x}$

$$\text{LHS}: \vec{x}'(t) = \frac{4}{5}(-1)e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{13\cdot 4}{5}e^{4t}\begin{pmatrix}2\\3\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}-4/5\,e^{-t} + \frac{104}{5}e^{4t}\\ \frac{4}{5}e^{-t} + \frac{156}{5}e^{4t}\end{pmatrix}}_{\text{LHS of } \vec{x}' = \mathbb{A}\vec{x}}$$

$$\text{RHS}: = \begin{pmatrix}1 & 2\\3 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2\\3 & 2\end{pmatrix}\begin{pmatrix}\frac{4}{5}e^{-t} + \frac{26}{5}e^{4t}\\ -\frac{4}{5}e^{-t} + \frac{39}{5}e^{4t}\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot\left(\frac{4}{5}e^{-t} + \frac{26}{5}e^{4t}\right) + 2\left(-\frac{4}{5}e^{-t} + \frac{39}{5}e^{4t}\right)\\ 3\cdot\left(\frac{4}{5}e^{-t} + \frac{26}{5}e^{4t}\right) + 2\left(-\frac{4}{5}e^{-t} + \frac{39}{5}e^{4t}\right)\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{4}{5}e^{-t} + \frac{104}{5}e^{4t}\\ \frac{4}{5}e^{-t} + \frac{156}{5}e^{4t}\end{pmatrix} \checkmark$$

Test I.C.

$$\vec{x}(0) = \frac{4}{5}e^{-0}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{13}{5}e^{4\cdot 0}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}\frac{4}{5}+\frac{26}{5}\\ -\frac{4}{5}+\frac{39}{5}\end{pmatrix} = \begin{pmatrix}6\\7\end{pmatrix}$$

- Boss Format (parametric eqn format)

Typically the boss does not appriciate vector form

So the specific solution can be broken out

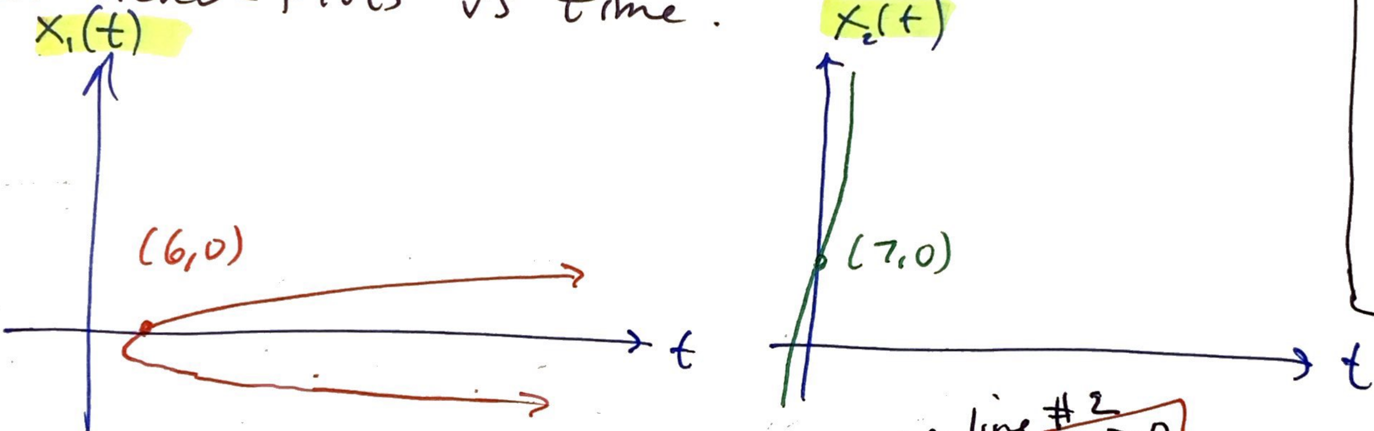
then $\vec{x}(t) = \frac{4}{5} e^{\overset{\lambda_1}{-t}} \underset{\vec{\eta}_1}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} + \frac{13}{5} e^{\overset{\lambda_2}{4t}} \underset{\vec{\eta}_2}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$

becomes

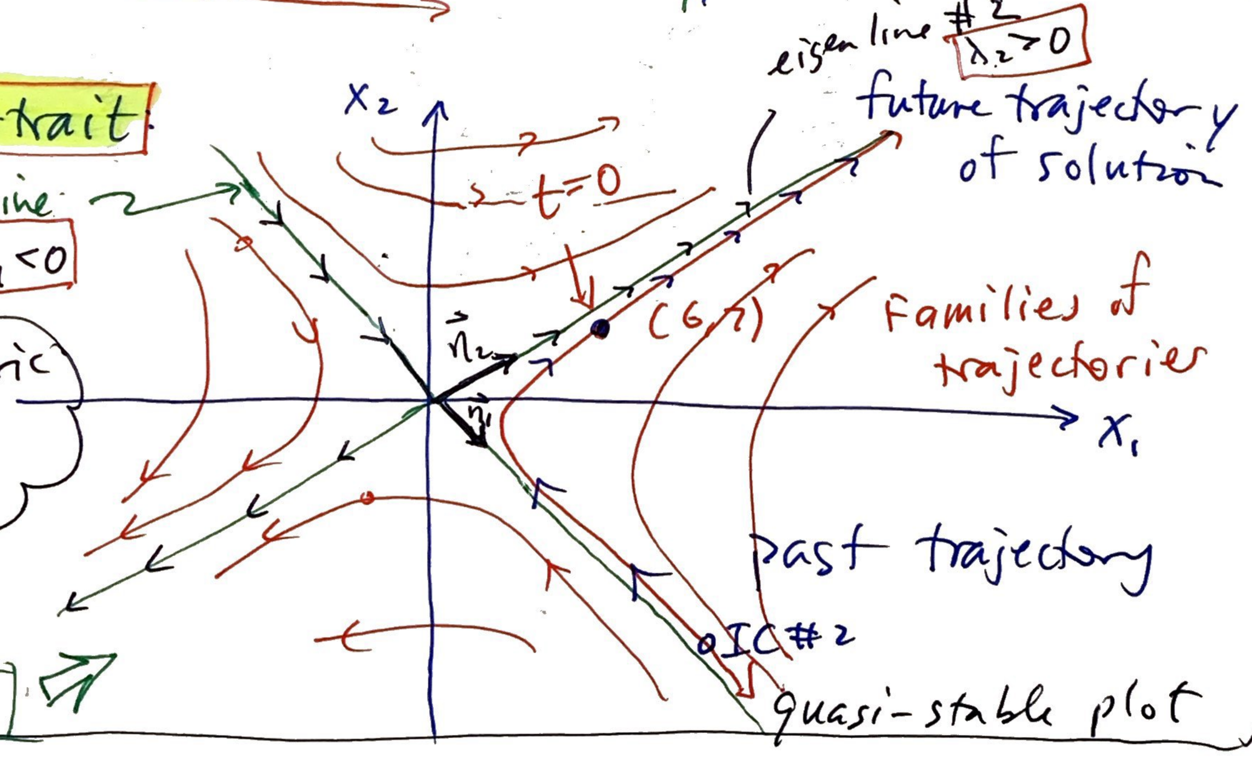
$$\boxed{\begin{aligned} x_1(t) &= \frac{4}{5} e^{-t} + \frac{26}{5} e^{4t} \\ x_2(t) &= -\frac{4}{5} e^{-t} + \frac{39}{5} e^{4t} \end{aligned}}$$

coupled scalar equations

- Individual plots vs time.



phase portrait



Project 4

Q: How do the eigenvectors manifest themselves on the x_1 vs. x_2 parametric plot we just looked at on Desmos?

• First some Observations {a pause}

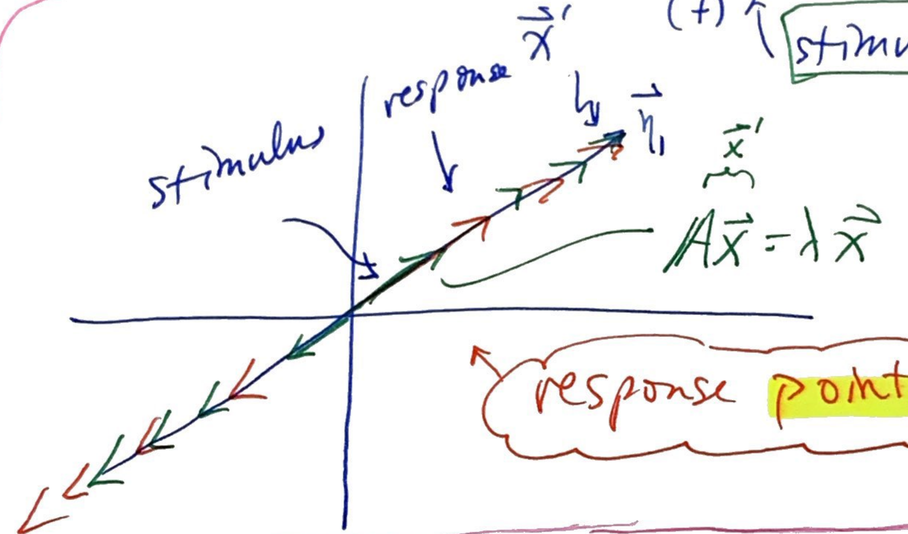
Recall $A\vec{x} = \lambda\vec{x}$ but $\vec{x}' = Ax$ so $\boxed{\vec{x}' = \lambda\vec{x}}$

input output

slope is of the e.vect. is ∥ to the eigenvector - SO - trajectories on the e.vector stay on the eigen.

• If $\boxed{\lambda > 0}$ then $\lambda\vec{x} = \vec{x}'$

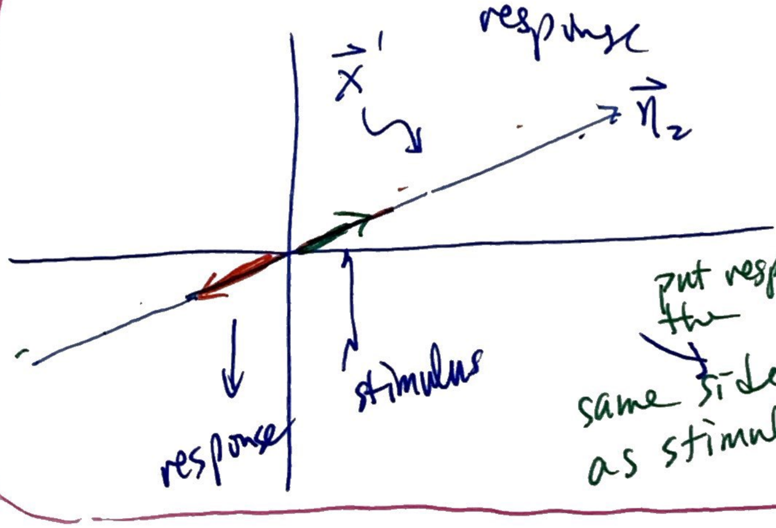
(+) stimulus response (slope)



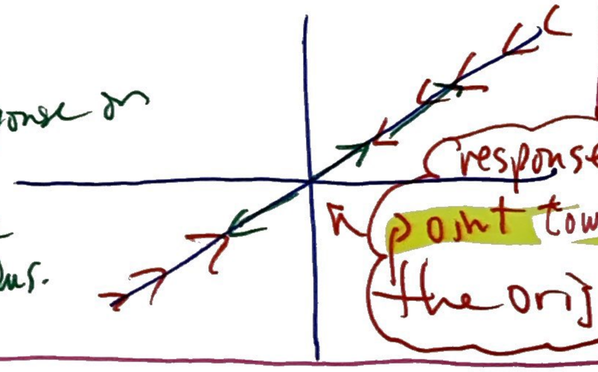
response point away from the origin

• If $\boxed{\lambda < 0}$ — $\boxed{\vec{x}' = \lambda\vec{x}}$

response stimulus } are in opposite directions



put response on the same side as stimulus.



responses point towards the origin

Comments:

1. If $\vec{x}_1(t)$ & $\vec{x}_2(t)$ are solutions to $\vec{x}' = \mathbb{A}\vec{x}$ then so is their linear combination

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

2. Define $\mathbb{X} \equiv \left(\vec{x}_1 \middle| \vec{x}_2 \middle| \cdots \middle| \vec{x}_n\right)$ where $\vec{x}_i' = \mathbb{A}\vec{x}_i$, i.e. $\mathbb{X}$ is colmns of solutions.

3. If $W \equiv \det(\mathbb{X})$ {wronskian} is non-zero then $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ are linearly indpendent solution vectors and we call the set The Fundamental Solutions of the syst. $\vec{x}' = \mathbb{A}\vec{x}$

4. If $W = 0$ we can try to eliminate some solution vectors and seek further solutions

{ If we fail to acheive n-solutions then seek more Lin. Indp. solutions }

Summary: Thm

$\boxed{\vec{x}_i \text{ are Lin. Independent}}$ iff $\boxed{\{x_i\}_{i=1}^{n} \text{ is Fundamental}}$ iff $\boxed{W \neq 0}$