7.4 Non-constant Coefficients w/ L.T.

- Recall: $f(t)$ is of exponential order if

$$\boxed{\lim_{s\to\infty} F(s) = 0 \quad \text{(L.T. Def)}}$$

Technically w/o L.T. ✓ math Def'n

$$\boxed{|f(t)| \le M e^{\alpha t} \quad \forall\, t \ge T}$$

for some α, M & T

"your function never out grows an exponential function"
We address simple ODE's with non-constant coeff:

- In the L.T. table, #30 we see combo's:

$$\boxed{\mathcal{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s) \qquad n = 1,2,3\ldots}$$

Like wise

$$\boxed{\mathcal{L}\left[e^{ct} f(t)\right] = F(s-c) \qquad \#29}$$

or

$$\boxed{\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(u)\,du \qquad \#31}$$

- These can be usefull in addressing non constant coefficients using the L.T.

Ex Solve the IVP

$$y'' + 3t\,y' - 6y = 2, \quad y(0)=0,\; y'(0)=0$$

(i) $\mathcal{L}$[ODE]

$$\mathcal{L}[y''] + 3\mathcal{L}[ty'] - 6\mathcal{L}[y] = \mathcal{L}[2]$$

??

rule #30 $\boxed{\mathcal{L}[t f(t)] = -1 \cdot F'(s)}$

let $f = y'$: $\mathcal{L}[t\,y'] = -\frac{d}{ds}\left[\mathcal{L}[y']\right]$

$$= -\frac{d}{ds}\left[(sY(s)) - y(0)\right]$$

product rule

$$= -\left[s'Y + sY'\right] + [y(0)]'$$

with $s' = 1$ and $[y(0)]' = 0$

- proceeding on w/ the transform of the ODE ...

$$\left[s^2 Y - sy(0) - y'(0)\right] + 3\left[-Y - sY'\right] - 6Y = \frac{2}{s}$$

our transformed eqn is not an algebra eqn as was to date ... it is an ODE itself.

$$\Rightarrow (s^2 - 3 - 6)Y - s\cdot 0 - 0 - 3sY' = \frac{2}{s}$$

$$\Rightarrow (s^2 - 9)Y - 3sY' = \frac{2}{s}$$

$$\Rightarrow \frac{-3s\,Y'}{-3s} + \frac{(s^2-9)Y}{-3s} = \frac{2}{s(-3s)}$$

Ex (cont.)

(ii) $\boxed{Y' - \underbrace{\left(\frac{s}{3} - \frac{3}{s}\right)}_{P(t)} Y = -\frac{2}{3s^2}}$ 1st order ODE in $Y(s)$

- This requires an integrating factor $\mu(s) = e^{\int p\,dt}$

$$\mu(s) = e^{\int\left(\frac{3}{s} - \frac{s}{3}\right)ds}$$

$$= e^{\left(3\ln(s) - \frac{1}{3}\frac{s^2}{2}\right)}$$

$$= e^{\ln s^3} e^{-s^2/6}$$

$$\boxed{\mu(s) = s^3 e^{-s^2/6}}$$

multiply through the ODE by μ yeild

$$\underbrace{s^3 e^{-s^2/6} Y' + s^3 e^{-s^2/6}\left(\frac{3}{s} - \frac{s}{3}\right)Y}_{} = s^3 e^{-s^2/6}\left(\frac{-2}{3s^2}\right)$$

$$\left(s^3 e^{-s^2/6} Y\right)' = -\frac{2}{3} s e^{-s^2/6}$$

integrate

$u = -\frac{s^2}{6}$

$du = -\frac{2s\,ds}{6}$

$$\int \left(s^3 e^{-s^2/6} Y\right)' ds = -\frac{2}{3}\int s e^{-s^2/6} ds$$

$$\Rightarrow \quad s^3 e^{-s^2/6} Y = \frac{2}{\cancel{3}}\int e^u du \cdot \cancel{3}$$

$$s^3 e^{-s^2/6} Y = 2e^{-s^2/6} + C$$

Solve for $Y(s)$

$$\boxed{Y(s) = \frac{2}{s^3} + c\,\frac{e^{s^2/6}}{s^3}}$$

(iii)

- We now need the inverse transform...

we can handle $\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = \frac{2}{2!}\mathcal{L}^{-1}\left(\frac{2!}{s^{2+1}}\right) = \frac{\cancel{2}}{\cancel{2!}}t^2$

$$y(t) = t^2 + c\,\mathcal{L}^{-1}\left[\frac{e^{s^2/6}}{s^3}\right]$$

- If $y(t)$ is assumed to be of exponential order, then per the opening definition,

$$\lim_{s\to\infty}\left(\frac{2}{s^3} + c\,\frac{e^{s^2/6}}{s^3}\right) = 0$$

$$= 0 + \boxed{c}\,\lim_{s\to\infty}\left(\frac{e^{s^2/6}}{s^3}\right) \neq 0$$

L'Hospital used 3-times

- The only way this is satisfied is if $\boxed{\underline{c=0}}$

Then we are of exponential order in y.

$$Y(s) = \frac{2}{s^2}$$

$$\boxed{y(t) = t^2}$$ since $c=0$

(6)

(ii) $\bar{Y}' + \frac{2}{s}\bar{Y} = \frac{2}{s^2}$

$$\mu = e^{\int \frac{2}{s} ds} = e^{2\ln(s)} = e^{\ln s^2} = s^2$$

mult. μ through the ODE in $\bar{Y}$:

$$\left(s^2\bar{Y}' + 2s\bar{Y}\right) = 2$$

$$\left(s^2\bar{Y}\right)' = 2$$

integrate

$$s^2\bar{Y} = \int 2\,ds$$

$$s^2\bar{Y} = 2s + C$$

$$\boxed{\bar{Y}(s) = \frac{2}{s} + \frac{c}{s^2}}$$

Q: how do we get "c" all well behaved.

$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

(iii) • the inverse

$$\boxed{y(t) = 2 + ct}$$

• Now apply $y(0) = 2$

$$\Rightarrow \quad y(0) = 2 + c(0)$$

$$2 = 2 + 0 \quad \text{No New info}$$

• Use y': if $y = 2 + ct$

$y' = 0 + c$ @ $t = 0$ $y' = -4$

$$\boxed{-4 = c}$$

Soln: $\boxed{y(t) = 2 - 4t}$ answer!

ove the IVP by . .

$$t y'' - t y' + y = 2, \quad y(0) = 2, \quad y'(0) = -4$$

- We know $\boxed{\mathcal{L}[ty'] = -sY' - Y}$ $y(0) = \text{const.}$

- likewise $\mathcal{L}[ty''] = -\frac{d}{ds}\left[\mathcal{L}[y'']\right]$ #30 $n=1$

$$= -\frac{d}{ds}\left[s^2 Y - s y(0) - y'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2 Y - s \cdot y_0 - (y')_0\right]$$

constant

$$= -\frac{d}{ds}\left[s^2 Y - s y_0 - (y')_0\right]$$

$$= -\left[(s^2 Y)' - y_0 - 0\right]$$

$y'(0) = \text{const}$

$$\boxed{\mathcal{L}[ty''] = -s^2 Y' - 2sY + y(0)}$$

- The ODE transforms to

(i) $\mathcal{L}[ty''] - \mathcal{L}[ty'] + \mathcal{L}[y] = \mathcal{L}[2]$

$$\left[-s^2 Y' - 2sY + y(0)\right] - \left[-sY' - Y\right] + Y = \frac{2}{s}$$

$y(0)$

$$\Longrightarrow (s - s^2) Y' + (2 - 2s) Y + 2 = \frac{2}{s}$$

$$\Longrightarrow \boxed{Y' + \frac{2}{s} Y = \frac{2}{s^2}}$$