7.2 Inverse Transforms & Solving ODE's

⊛ Transforming derivatives and Integrals

Before we approach ODE's lets introduce eqns that have an unknown function, it's derivative AND an integral of it. We call these integral-diff'tl eqns. We use the following: formula #32

$$\boxed{\mathcal{L}\left[\int_0^t f(v)\,dv\right] = \frac{F(s)}{s}}$$

• An Integral-diff. eqn

Ex
$$y' + \int_0^t y(v)\,dv + y = t^2$$

$\boxed{\text{I}}$ one soln method → diff't both sides

$$\frac{d}{dt}\left[y' + \int_0^t y(v)\,dv + y = t^2\right]$$

$$\Rightarrow \quad y'' + \underbrace{\frac{d}{dt}\int_0^t y(v)\,dv}_{\text{F. Thm. of Calc.} \Rightarrow y(t)} + y' = 2t$$

$$\Rightarrow \quad y'' + y + y' = 2t$$

$$\Rightarrow \quad \boxed{y'' + y' + y = 2t}$$

pure ODE
we can solve this.

OR $\boxed{\text{II}}$ Use Laplace Transforms and formula 32.
(we will not cover this but you may see it the HW)

(*) Pure ODE's

- When we solve an ODE we will transform the entire ODE. This means we need to know how to handle this

$$\mathcal{L}[y']$$

- Using the definition, $\mathcal{L}[f'] = \int_0^\infty e^{-st} f'(t)\,dt$,

we have

$$\mathcal{L}[f'(t)] = \lim_{T\to\infty}\left[\int_0^T \underbrace{e^{-st}}_{u}\ \underbrace{f'(t)\,dt}_{dv}\right]$$

e^{-st}	$-se^{-st}dt$
$f'dt$	f

- use parts, $\int u\,dv = uv - \int v\,du$, we get:

$$\mathcal{L}[f'(t)] = e^{-st}\cdot f(t)\Big|_0^T - (-s)\int_0^T e^{-st} f(t)\,dt$$

$$= \left\{\lim_{T\to\infty} e^{-sT} f(T) - f(0)\right\} + s\cdot\lim_{T\to\infty}\int_0^T e^{-st} f(t)\,dt$$

- clean up $= 0^{*} - f(0) + s\,\mathcal{L}[f(t)]$

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$

- So $\mathcal{L}[\text{ODE}]$ will result in an ALGEBRAIC equation in $Y(s)$.

I.C.

$$\boxed{\mathcal{L}[y'] = sY(s) - \overset{\text{I.C.}}{y(0)}}$$ ← formula #35

* $f(T)$ needs to be of exponential order: $f(t) < e^{\alpha t}$ $\forall \alpha > 0$

EX Solve $y' + y = t$, $y(0) = 1$

(i) $\mathcal{L}[\text{ODE}]$: $\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[t]$

$$\left(sY - y(0)\right) + Y = \frac{1}{s^2}$$

$$sY - 1 + Y = \frac{1}{s^2}$$

factor Y:

$$(s+1)Y = \frac{1}{s^2} + 1$$

Solve for Y:

$$Y(s) = \frac{\frac{1}{s^2}+1}{s+1}$$

(ii)

$$\boxed{Y(s) = \frac{s^2+1}{s^2(s+1)}}$$

algebra!

(iii) apply the inverse transform to solve

$$\boxed{y(t) = \mathcal{L}^{-1}\left[\frac{s^2+1}{s^2(s+1)}\right]}$$

Q: How do we get $\frac{s^2+1}{s^2(s+1)}$ into a form where we can do an inverse transform, i.e. be able read the table in reverse?

* We pause & practice then resume on p 8

✱ Inverse Transforms

Practice on simple functions first.

EX If $F(s) = \dfrac{19}{s+2} - \dfrac{1}{3s-5} + \dfrac{7}{s^2}$ Find $f(t)$

$$\mathcal{L}^{-1}\left[F(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^2}\right]$$

$$f(t) = 19\,\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{3(s-5/3)}\right] + 7\,\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

#2 with $a=-2$

#2 with $a = 5/3$

#3 with $n=1$

$$\boxed{f(t) = 19 \cdot e^{-2t} - \frac{1}{3}e^{\frac{5}{3}t} + 7 \cdot t}$$

EX $F(s) = \dfrac{6s}{s^2+25} + \dfrac{3}{s^2+25}$. Find $f(t)$

$$f(t) = 6\,\mathcal{L}^{-1}\left[\frac{s}{s^2+5^2}\right] + \frac{3}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]$$

#8

#7

$$\boxed{f(t) = 6\cos(5t) + \frac{3}{5}\sin(5t)}$$

* polynomials

Ex If $F(s) = 11/s^5$ Find $f(t)$

$$\#3: \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

Use $n = 4$

i.e. $\boxed{\mathcal{L}^{-1}\left[\frac{4!}{s^{4+1}}\right] = t^4}$

So $\mathcal{L}^{-1}\left[\frac{11}{s^5}\right]$

$$= \frac{11}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^{4+1}}\right]$$

Thus...

$$f(t) = \frac{11}{4!}t^4 \quad \text{or} \quad \boxed{f(t) = \frac{11}{24}t^4}$$

Ex $F(s) = \frac{6s-5}{s^2+7}$ Find $f(t)$

$$F(s) = 6\left[\frac{s}{s^2+(\sqrt{7})^2}\right] - \frac{5}{\sqrt{7}}\left[\frac{1\cdot\sqrt{7}}{s^2+(\sqrt{7})^2}\right]$$

$$f(t) = 6\mathcal{L}^{-1}\left[\frac{s}{s^2+\sqrt{7}^2}\right] - \frac{5}{\sqrt{7}}\mathcal{L}^{-1}\left[\frac{\sqrt{7}}{s^2+\sqrt{7}^2}\right]$$

$$\boxed{f(t) = 6\cos(\sqrt{7}t) - \frac{5}{\sqrt{7}}\sin(\sqrt{7}t)}$$

EX If $F(s) = \dfrac{1}{s(s+1)}$ Find $f(t)$

- Recall our partial fractions from pre-calculus...
  we can decompose $\dfrac{1}{s(s+1)}$ into the sum of fractions

i.e.
$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\frac{1}{s(s+1)} = \frac{A(s+1) + sB}{s(s+1)}$$

- Since we have the same denominator
  we can equate the numerator

$$1 = A(s+1) + sB$$

- Then we match powers of "s" on each side

$$s^1 : \; 0 = A + B \longrightarrow B = -1$$
$$s^0 : \; 1 = A$$

So
$$\boxed{\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}}$$

- Now
$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

#1   #2 with $a=-1$

$$= 1 - e^{-t}$$

$$\boxed{f(t) = 1 - e^{-t}}$$

⊛ Partial Fraction Decompositions Forms:

$$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

$$\frac{1}{(s+a)^2(s+b)} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$$

$$\frac{1}{(s^2-2s-4)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s-4}$$

$$\frac{1}{(s+1)(s^2-2s-4)^2} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s-4} + \frac{Ds+F}{(s^2-2s-4)^2}$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s}\cdot\frac{s}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

-OR-

$$\frac{1}{(s^2)(s+1)} = \frac{As+B}{(s^2)} + \frac{C}{s+1}$$

EX (cont.) We can now Finish solving the ODE from our first example

$$\boxed{y' + y = t \quad , \quad y(0) = 1}$$

$$\mathcal{L}[\;] \Rightarrow Y(s) = \frac{1+s^2}{s^2(s+1)}$$

So $y(t) = \mathcal{L}^{-1}\left[\dfrac{1+s^2}{s^2(s+1)}\right]$

• Lets decompose

$$\frac{1+s^2}{(s^2)(s+1)} = \frac{As+B}{s^2} + \frac{C}{s+1}$$

$$= \frac{(As+B)(s+1) + Cs^2}{(s^2)(s+1)}$$

• equate Numerators

$$1 + s^2 = As^2 + Bs + As + B + Cs^2$$

• equate powers

s^2 : $1 = A + C \rightarrow C = 2$

s^1 : $0 = B + A \rightarrow A = -1$

s^0 : $1 = B$

• State the decomposition

$$\Rightarrow \frac{1+s^2}{s^2(s+1)} = \frac{-s+1}{s^2} + \frac{2}{s+1}$$

• Perform the inverse transform

(iii) $\mathcal{L}^{-1}$

$$\boxed{Y(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{2}{s+1}}$$

Reverse LookUp

$$\boxed{y(t) = -1 + t + 2e^{-t}}$$

Test it! It works!

* Second derivatives

$$\boxed{\mathcal{L}[y''] = s^2 Y(s) - s\,y(0) - y'(0)}$$ formula #36

Ex Solve $y'' + 9y = 0$ if $y(0)=1$, $y'(0)=0$

(i) $\mathcal{L}[\text{ODE}] : \mathcal{L}[y''] + 9\mathcal{L}[y] = 0$

#36

$$\left(s^2 Y - s\,y(0) - y'(0)\right) + 9 \cdot Y = 0$$

$$\left(s^2 Y - s\cdot 1 - 0\right) + 9Y = 0$$

(ii) $(s^2+9)Y = s \Rightarrow \boxed{Y(s) = \dfrac{s}{s^2+9}}$

(iii) $y(t) = \mathcal{L}^{-1}[Y(s)]$

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+3^2}\right]$$

$\boxed{y(t) = \cos(3t)}$ Soln.

EX Solve $y'' + 9 = 0$, $y(0) = 1$, $y'(0) = 0$

No y this time

(i) $\mathcal{L}[\text{ODE}] \rightarrow \left(s^2 Y - s\,\overset{1}{y(0)} - \overset{0}{y'(0)}\right) + \left(9/s\right) = 0$

$\rightarrow \quad s^2 Y - s + \frac{9}{s} = 0$

$$Y(s) = \frac{s}{s^2} + \frac{9}{s \cdot s^2}$$

(ii) $\boxed{Y(s) = \frac{1}{s} - \frac{9}{s^3}\frac{2!}{2!}}$ #3 w/ $n = 2$

(iii) $y(t) = 1 - \frac{9t^2}{2!}$

$\boxed{y(t) = -\frac{9}{2}t^2 + 1}$ Soln of I.V.P.

Ex Solve $y'' - 10y' + 9y = 5t$; $y(0) = -1$ & $y'(0) = 2$

(i) $\mathcal{L}[y'' - 10y' + 9y] = \mathcal{L}[5t]$

$$\Rightarrow \left[s^2 Y - s\,y(0) - y'(0)\right] - 10\left[sY - y(0)\right] + 9Y = 5/s^2$$

$$\left[s^2 Y - s(-1) - (2)\right] - 10sY + 10\cdot(-1) + 9Y = \frac{5}{s^2}$$

(ii) Solve for Y

$$(s^2 - 10s + 9)Y = -s + 2 + 10 + \frac{5}{s^2}$$

$$Y = \frac{5/s^2 - s + 12}{s^2 - 10s + 9} \cdot \frac{s^2}{s^2}$$

$$Y = \frac{5 - s^3 + 12s^2}{s^2(s^2 - 10s + 9)}$$

$$\boxed{Y = \frac{-s^3 + 12s^2 + 5}{s^2(s-1)(s-9)}}$$

- Partial Fractions

$$Y = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$Y = \frac{As(s-9)(s-10) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)}{s^2(s-9)(s-1)}$$

- equate numerators: $\boxed{\text{pick strategic numbers}}$

$$5 - s^3 + 12s^2 = As(s-9)(s-1) + B(s-9)(s-1) + C(s^2)(s-1) + Ds^2(s-9)$$

- $s=0$: $5 - 0^3 + 12\cdot 0^2 = A\cdot 0\cdot(0-9)(0-1) + B(0-9)(0-1) + C\cdot 0^2\cdot(0-1) + D\cdot 0^2(0-9)$

$$\Rightarrow \quad 5 = B(-9)(-1) \longrightarrow \boxed{B = 5/9}$$

- $s=9$: $5 - 9^3 + 12\cdot 9^2 = A\cdot 0 + B\cdot 0 + C\cdot 9^2(9-1) + D\cdot 0$

$$248 = C(648) \longrightarrow \boxed{C} = \frac{248}{648} = \boxed{\frac{31}{81}}$$

- $s=1$: $5-1^3+12\cdot 1^2 = A\cdot 0 + B\cdot 0 + C\cdot 0 + D\cdot 1^2\cdot(1-9)$

$$16 = D(-8) \rightarrow D = \frac{16}{-8} \Rightarrow \boxed{D=-2}$$

Q: What do we do for A: ? $s=0$ has been used

A: Pick any number: $s=-1$

- $s=-1$: $5-(-1)^3+12(-1)^2 = \underbrace{A}_{\text{only unknown}}(-1)(-1-9)(-1-1) + \overbrace{\left(\frac{5}{9}\right)}^{B}(-1-9)(-1-1) + \overbrace{\left(\frac{31}{81}\right)}^{C}(-1)^2(-1-1) + \underbrace{(-2)}_{D}(-1)^2(-1-9)$

$$5+1+12 = -20A + \frac{100}{9} - \frac{62}{81} + 20$$

$$18 = -20A + \frac{900-62+81\cdot 20}{81}$$

$$18 = -20A + \frac{2458}{81}$$

$$\frac{\overbrace{18\cdot 81 - 2458}^{1000}}{81} = -20A \rightarrow \boxed{A = \frac{\cancel{1000}^{50}}{\cancel{20}\cdot 81} = \frac{50}{81}}$$

So the decomposed fraction is

$$\boxed{Y(s) = \frac{50/81}{s} + \frac{5/9}{s^2} + \frac{31/81}{s-9} - \frac{2}{s-1}}$$

(iii) $\mathcal{L}^{-1}\left[Y(s) = \frac{50}{81}\cdot\frac{1}{s} + \frac{5}{9}\frac{1}{s^2} + \frac{31}{81}\cdot\frac{1}{s-9} - 2\cdot\frac{1}{s-1}\right]$

$$\boxed{y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t}$$ Soln

Ex Solve $y'' - 6y' + 15y = 2\sin(3t)$, $y(0) = -1$, $y'(0) = 4$

(i) $\mathcal{L}[\text{ODE}]$

$$\left(s^2 Y - s y(0) - y'(0)\right) - 6\left(sY - y(0)\right) + 15Y = 2 \cdot \frac{3}{s^2+3^2}$$

(ii) Solve for Y:

$$(s^2 - 6s + 15)Y = s\overset{y(0)}{(-1)} + \overset{y'(0)}{(-4)} - 6\overset{y(0)}{(-1)} + \frac{6}{s^2+9}$$

$$Y(s) = \frac{(-s+2)^{*}\frac{s^2-9}{s^2-9} + \frac{6}{s^2+9}}{(s^2-6s+15)}$$

$$Y = \frac{(-s+2)(s^2-9) + 6}{(s^2+9)(s^2-6s+15)}$$

$$\boxed{Y(s) = \frac{-s^3+2s^2-9s+24}{(s^2+9)(s^2-6s+15)}}$$

- decompose $Y(s) = \dfrac{As+B}{s^2+9} + \dfrac{Cs+D}{s^2-6s+15}$

$$= \frac{(As+B)(s^2-6s+15) + (Cs+D)(s^2+9)}{(s^2+9)(s^2-6s+15)}$$

- equate numerators: { FOIL and Factor out "s" powers }

$$-s^3 + 2s^2 - 9s + 24 = (A+C)s^3 + (-6A+B+D)s^2 + (15A-6B+9C)s + 15B + 9D$$

⋮

$\boxed{Ex}$ cont.

- equate powers

$$s^3 : -1 = A + C$$
$$s^2 : 2 = -6A + B + D$$
$$s^1 : -9 = 15A - 6B + 9C$$
$$s^0 : 24 = 15B + 9D$$

$\boxed{4\times4}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ -6 & 1 & 0 & 1 \\ 15 & -6 & 9 & 0 \\ 0 & 15 & 0 & 9 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -9 \\ 24 \end{pmatrix}$$

- Solving yeilds : $\boxed{A = \frac{1}{10},\ B = \frac{1}{10},\ C = \frac{-11}{10},\ D = \frac{25}{10}}$

{see attached pages}

- Plug A, B, C, D into the decomposition Form

$$Y(s) = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 - 6s + 15}$$

New Technique

$$\boxed{Y = \frac{1}{10}\left(\frac{s+1}{s^2+9}\right) + \frac{1}{10}\left(\frac{-11s + 25}{s^2 - 6s + 15}\right)}$$

- Complete the square : $s^2 - 6s + 15 = (s-3)^2 + 6$

$$Y(s) = \frac{1}{10}\left(\frac{s}{s^2+3^2}\right) + \frac{1}{3\cdot 10}\left(\frac{3}{s^2+3^2}\right) + \frac{1}{10}\left(\frac{-11(s-3+3) + 25}{(s-3)^2 + \sqrt{6}^2}\right)$$

$$Y(s) = \frac{1}{10}\left(\frac{s}{s^2+3^2}\right) + \frac{1}{30}\left(\frac{3}{s^2+3^2}\right) - \frac{11}{10}\left(\frac{s-3}{(s-3)^2 + \sqrt{6}^2}\right) - \frac{8}{\sqrt{6}\,10}\left(\frac{\sqrt{6}}{(s-3)^2 + \sqrt{6}^2}\right)$$

$$y(t) = \frac{1}{10}\cos(3t) + \frac{1}{30}\sin(3t) - \frac{11}{10}\left[e^{3t}\cos(\sqrt{6}t)\right] - \frac{8}{10\sqrt{6}}e^{3t}\sin(\sqrt{6}\,t)$$

formula #20 #19

* So far we have seen two methods used to determine A, B, C, D ... in partial fraction decomposition: Full on matrix soln & strategaic numbers.

- There is yet a 3rd method that can be used, referred to as Heavyside's 1st method. Attached are two examples.

- This method will not produce results on denominators with s^2 or higher terms use one of the 1st two methods we discussed.

- Finally one more complete the square approach is presented.

ex
$$G(s) = \frac{86s-78}{(s+3)(s-4)(5s-1)}$$

$$G(s) = \frac{A}{s+3} + \frac{B}{s-4} + \frac{C}{5s-1}$$

Since all denom's are linear to power 1, we can use

- Heavyside method...

$$A = \frac{86s-78}{\cancel{(s+3)}(s-4)(5s-1)}\Bigg|_{s=-3} = \frac{86(-3)-78}{(-3-4)(5(-3)-1)} = \frac{-336}{7\cdot 16} = \boxed{-3}$$

$$B = \frac{86s-78}{(s+3)\cancel{(s-4)}(5s-1)}\Bigg|_{s=4} = \frac{86(4)-78}{(4+3)(5\cdot 4-1)} = \frac{266}{7\cdot 19} = \boxed{2}$$

$$C = \frac{86s-78}{(s+3)(s-4)\cancel{(5s-1)}}\Bigg|_{s=\frac{1}{5}} = \frac{86(\frac{1}{5})-78}{(\frac{1}{5}+3)(\frac{1}{5}-4)} = \frac{\frac{86-390}{5}}{\frac{16}{5}\cdot(-\frac{19}{5})} = \frac{1520}{16\cdot 19} = \boxed{5}$$

So

$$G(s) = -3\left(\frac{1}{s+3}\right) + 2\left(\frac{1}{s-4}\right) + \frac{\cancel{5}\,1}{\cancel{5}}\left(\frac{1}{s-1/5}\right)$$

$\mathcal{L}^{-1}$

$$\boxed{g(t) = -3e^{-3t} + 2e^{4t} + e^{\frac{1}{5}t}}$$

EX $H(s) = \dfrac{s+7}{s^2-3s-10}$, Find $\mathcal{L}^{-1}[H]$

Note that the denom. factors nicely!

$$H(s) = \frac{s+7}{(s+2)(s-5)}$$

choices : complete square OR Partial Fractions (prefered when you can factor nicely)

Form:

$$\frac{s+7}{(s+2)(s-5)} = \frac{A}{s+2} + \frac{B}{s-5}$$

$\dfrac{A}{(s+2)=0}$

Use Heavysides Method:

$$A = \frac{s+7}{\cancel{(s+2)}(s-5)}\Bigg|_{s=-2} = \frac{-2+7}{-2-5} = -\frac{5}{7}$$

$$B = \frac{s+7}{(s+2)\cancel{(s-5)}}\Bigg|_{s=5} = \frac{5+7}{(5+2)} = \frac{12}{7}$$

So

$$H(s) = -\frac{5}{7}\left(\frac{1}{s+2}\right) + \frac{12}{7}\left(\frac{1}{s-5}\right)$$

$\mathcal{L}^{-1}$

$$\boxed{h(t) = -\frac{5}{7}e^{-2t} + \frac{12}{7}e^{5t}}$$

Another Complete The Square Approach

EX Find $f(t)$ if $F(s) = \dfrac{1-3s}{s^2+8s+21}$

(i) match denominator ; we need to complete the square

$$s^2+8s+\underbrace{\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2}_{\text{magic zero}}+21$$

$$= \overbrace{\left(s+\frac{8}{2}\right)^2} - 4^2+21$$

$$= (s+4)^2+5$$

~~Then we get~~

SKIP

$$\boxed{F(s) = \frac{1-3s}{(s+4)^2+5}} = \cancel{\frac{1}{(s+4)^2+5} - 3\cdot\frac{s}{(s+4)^2+5}}$$

⊛ don't split up yet!! premature

(ii) work on numerators... apply shift in the numerator.

$$= \frac{1-3(s+4-4)}{(s+4)^2+5}$$

now split

$$= \frac{1-3(s+4)+12}{(s+4)^2+(\sqrt{5})^2} = -3\frac{s+4}{(s+4)^2+(\sqrt{5})^2} + 13\frac{s+4}{(s+4)^2+\sqrt{5}^2}$$

e^{bt} $\cos\sqrt{5}t$

$$F(s) = -3\left(\frac{s-(-4)}{(s-(-4))^2+\sqrt{5}^2}\right) + \frac{13}{\sqrt{5}}\cdot\left(\frac{1\cdot\sqrt{5}}{(s-(-4))^2+\sqrt{5}^2}\right)$$

$\mathcal{L}^{-1}$ Formula

$$\boxed{f(t) = -3e^{-4t}\cos(\sqrt{5}t) + \frac{13}{\sqrt{5}}\cdot e^{-4t}\sin(\sqrt{5})}$$