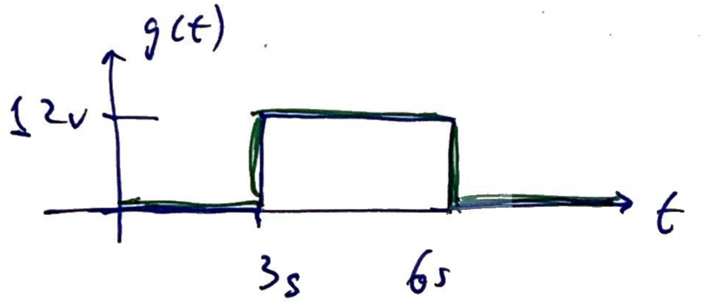
Chapter 7 The Laplace Transform Method

- From Chpt 4 we learned to solve 2nd order ODE's with largely constant coefficients
- For non-homogeneous ODE's we explored
  - → Undetermined Coefficients (simple $g(t)$)
  - → Variation of Parameters (more complex $g(t)$)
- We now address a 3rd method called

  The Laplace Transform Method

  - The process converts an ODE into an algebra problem. It is not worth the effort if $g(t)$ is simple, but if the driving function is more complex the the LT method is preffered.

- Applications that have step-wise driving functions benefit from L.T. method.
  - → Electronics this would be turning on of a switch for 3 seconds after energizing a circuit, and then shutting it of 3s later

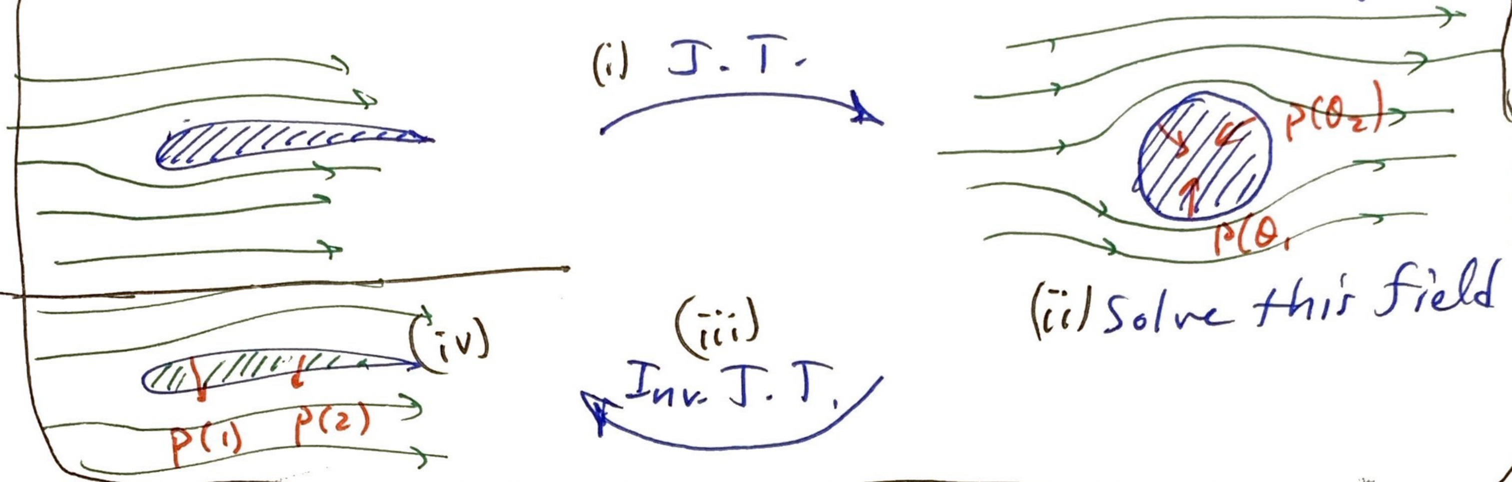


"shock to the system"

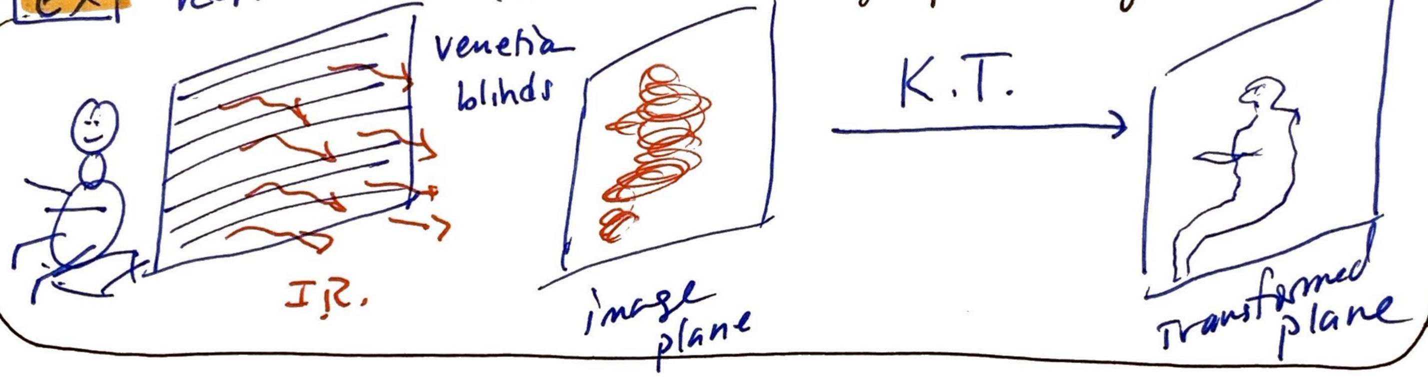
7.1 Intro to the L.T.

- Transforms convert a problem in one setting into a hopefully easier setting.

EX The Joukowski Transformation (aerodynamics)



EX kernel transformation (image processing)



EX MRI.

The Laplace Transform Method

(i) Transform the ODE

(ii) Solve the resulting algebra problem

(iii) Apply the inverse transform to get a solution

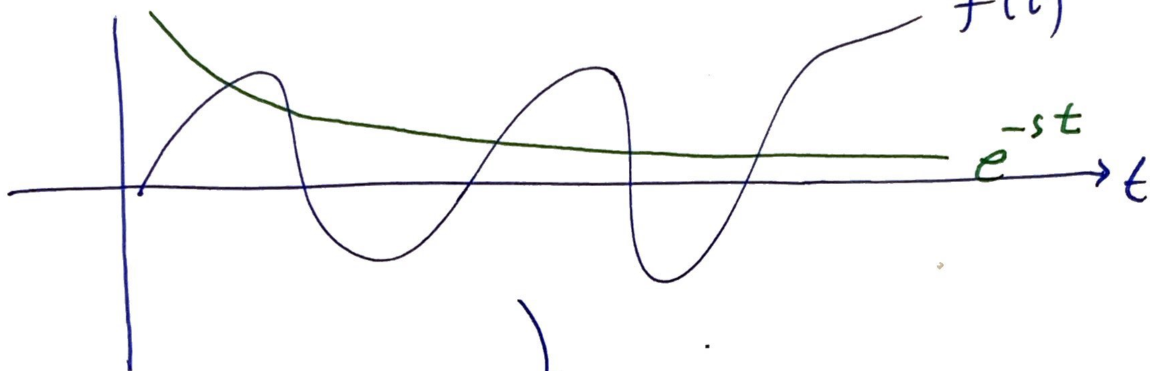
⊗ Definition

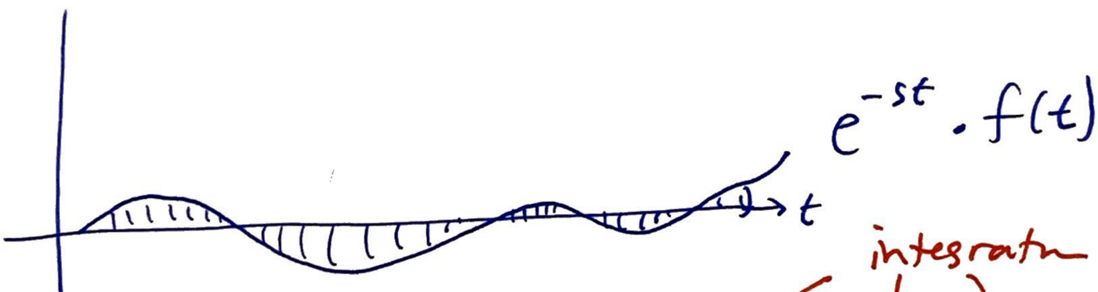
The Laplace Transform of a function $f(x)$ is defined to be

some function

$$\boxed{\mathcal{L}\left[f(t)\right] \equiv \int_0^\infty \boxed{e^{-st}} \cdot f(t)\, dt = F(s)}$$

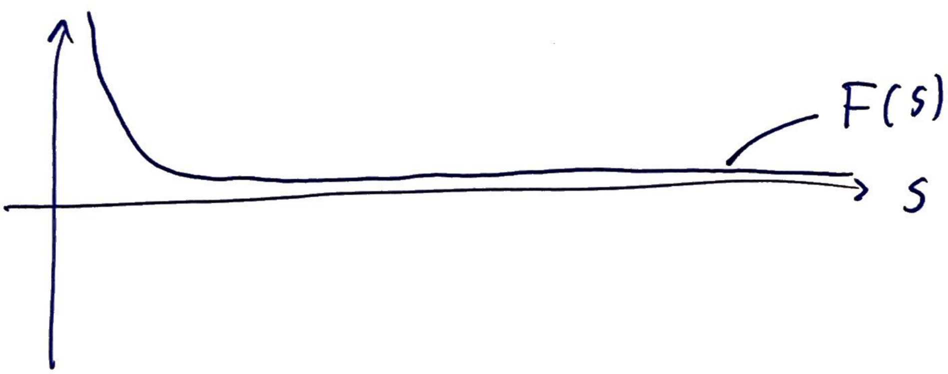
area amplitude





integration

$$\int_0^\infty e^{-st} \cdot f(t)\, dt = F(s)$$



⊛ Lets start learning some transforms

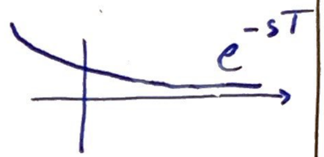
ex Find $\mathcal{L}[1]$

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\cdot 1 \cdot dt \qquad u = -st$$

$$= \lim_{T\to\infty}\left(-\frac{1}{s} e^{-st} \Big|_{t=0}^{T} \right)$$

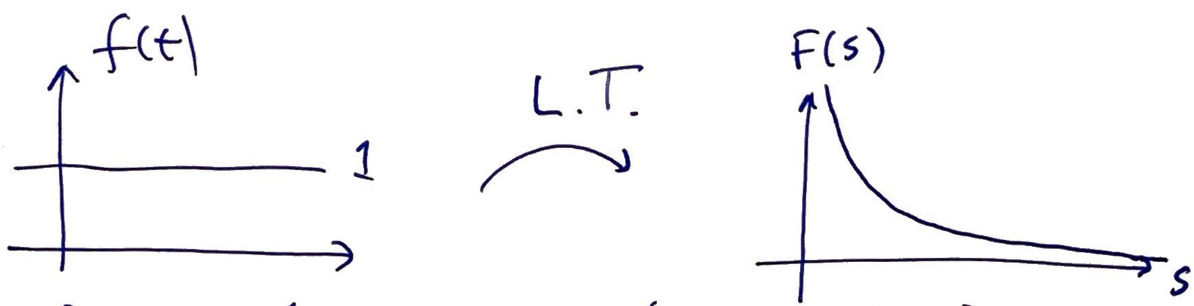
$$= \lim_{T\to\infty}\left(-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{-s\cdot 0} \right)$$

$$= \lim_{T\to\infty}\left(-\frac{1}{s} \cancel{e^{-sT}}^{\,\to 0} \right) + \frac{1}{s}$$



$$= \frac{1}{s}$$

So $\boxed{\mathcal{L}[1] = \frac{1}{s}}$ $\quad s > 0 \quad$ formula #1



we frequently ignore the conditions on "s" but we should remember the exist.

Ex Find $\mathcal{L}[e^{at}]$ $f(t)$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}[e^{at}]\,dt$$

$$= \int_0^\infty e^{(a-s)}\,dt \qquad u = a-s$$

$$= -\frac{1}{a-s} \quad \text{if} \quad a-s<0 \text{ or } s>a$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}}$$

formula #2

- We will build up a Library of these transforms and place them in a table of Laplace Transf

EX Find $\mathcal{L}[\sin(at)]$

$$\mathcal{L}[\sin(at)] \equiv \int_0^\infty e^{-st}[\sin(at)]\,dt$$

$$= \lim_{T\to\infty} \int_0^T e^{-st}\sin(at)\,dt$$

go to an integral table

$$= \lim_{T\to\infty} \left\{ \frac{e^{-st}}{s^2+a^2}(-\sin(at) - a\cos(at)) \right\}_0^T$$

$$= \frac{1}{s^2+a^2}\left[\left[\lim_{T\to\infty} \cancel{e^{-sT}}^{\,0}(\sin(Ta)+a\cos(Ta)\right] - \left[0 - a\cdot 1\right]\right]$$

$$= \frac{a}{s^2+a^2}, \quad s>0$$

Formula #7

$$\boxed{\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}} \quad s>0$$

we will find

$$\boxed{\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}}$$

formula #8

The L.T. is a linear transformation

$$\mathcal{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha F + \beta G$$

where $F = \mathcal{L}[f]$ & $G = \mathcal{L}[g]$

EX Find $\mathcal{L}[f]$ if $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$\mathcal{L}\left[6e^{-5t} + e^{3t} + 5t^3 - 9\right]$$

$$= 6\,\underbrace{\mathcal{L}[e^{-5t}]} + \mathcal{L}[e^{3t}] + 5\,\mathcal{L}[t^3] - 9\,\mathcal{L}[1]$$

L.T. Table

$$= 6\left(\frac{1}{s+5}\right) + \left(\frac{1}{s-3}\right) + 5\left(\frac{3!}{s^{3+1}}\right) - 9\left(\frac{1}{s}\right)$$

So

$$\boxed{F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}}$$

Ex Find $\mathcal{L}[f(t)]$ if $f(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$F(s) = 4\,\mathcal{L}[\cos(4t)] - 9\,\mathcal{L}[\sin(4t)] + 2\,\mathcal{L}[\cos(10t)]$$

$$= 4\left(\frac{s}{s^2+4^2}\right) - 9\left(\frac{4}{s^2+4^2}\right) + 2\left(\frac{s}{s^2+10^2}\right)$$

$$\boxed{F = \frac{4s-36}{s^2+16} + \frac{2s}{s^2+100}}$$

⊛ Lets examine an $f(t)$ that is a product of an exponential and a trig function

Ex If $g(t) = e^{3t}\cos(6t)$, Find $\mathcal{L}[g]$

Formula #20

$$\mathcal{L}[e^{at}\cos(bt)] = \frac{(s-a)}{(s-a)^2+b^2} = \left.\frac{s}{s^2+b^2}\right|_{s \to s-a}$$

$$\mathcal{L}[e^{\overset{a}{3}t}\cos(\overset{b}{6}t)] = \frac{(s-3)}{(s-3)^2+6^2}$$

Replace s with s−3

- In general formula #29

$$\boxed{\mathcal{L}\left[e^{ct} f(t)\right] = F(s-c)}$$

when $F(s) = \mathcal{L}\left[f(t)\right]$

Ex $\mathcal{L}\left[e^{3t} t^2\right]$

$= F(s-c)$

so here $F(s) = \mathcal{L}\left[t^2\right]$ Formula #3

$$= \frac{2!}{s^{2+1}}$$

$$F(s) = \frac{2}{s^3}$$

$$\boxed{\mathcal{L}\left[e^{3t} t^2\right] = \frac{2}{(s-3)^3}}$$

- In general formula #30

$$\boxed{\mathcal{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s)}$$

Ex $\mathcal{L}\left[t\cosh(3t)\right]$

#30

$$= (-1)^1 \left(\mathcal{L}\left[\cosh(3t)\right]\right)'$$

diff't

#18

$$= -1 \cdot \left(\frac{s}{s^2-3^2}\right)'$$

quotient rule

$$= -\left[\frac{(s)'(s^2-9) - s\cdot(s^2-9)'}{(s^2-9)^2}\right]$$

$$= -\frac{(s^2-9) - s\cdot 2s}{(s^2-9)^2}$$

$$= -\frac{s^2-2s^2-9}{(s^2-9)^2}$$

$$= \boxed{\frac{s^2+9}{(s^2-9)^2}}$$

7.1 #3 Find $\mathcal{L}[f]$ if $f(t)=\begin{cases} t & [0,1) \\ 1 & [1,\infty) \end{cases}$

- Apply the definition:

$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\, t\, dt + \int_1^\infty e^{-st}\cdot 1 \cdot dt$$

Table of Integrals $\boxed{\int u e^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C}$

So

- $$\int_0^1 e^{-st}\, t\, dt = \frac{1}{(-s)^2}(-st-1)e^{-st}\Big|_0^1 = \frac{(-s\cdot 1-1)e^{-s\cdot 1}}{s^2} - \frac{(0-1)e^0}{s^2}$$

$$= -\frac{(s+1)e^{-s}}{s^2} + \frac{1}{s^2}$$

Table $$\int e^{-st}\,dt = \int e^u \frac{du}{-s} = \frac{e^u}{-s} = \frac{e^{-st}}{-s} + c$$

$u=-st$
$du=-s\,dt$

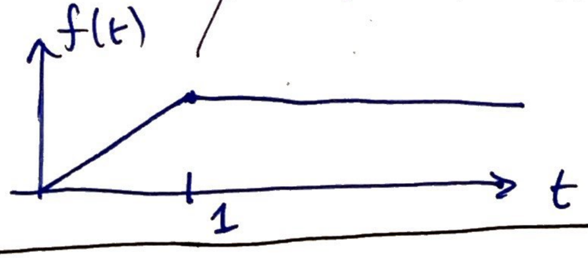
So

- $$\int_1^\infty e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_1^\infty = \frac{0}{-s} - \frac{e^{-s\cdot 1}}{-s} = \frac{e^{-s}}{s}$$

- Together

$$\mathcal{L}[f] = \left(-\frac{(s+1)e^{-s}}{s^2} + \frac{1}{s^2}\right) + \left(\frac{e^{-s}}{s}\right)$$

$$F(s) = \cancel{-\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \cancel{\frac{e^{-s}}{s}} = \boxed{\frac{1-e^{-s}}{s^2} = F(s)}$$



Ex Find $\mathcal{L}[t^{3/2}]$ { mymathmantra.com → ODE resource → Laplace Trans. Table }

I The easy way: Formula #6 $\boxed{\mathcal{L}[t^{n-\frac{1}{2}}] = \frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+1/2}}}$

n = 2 :

$$\mathcal{L}[t^{2-1/2}] = \frac{1\cdot(2\cdot 2-1)\sqrt{\pi}}{2^2 s^{2+1/2}} = \boxed{\frac{3\sqrt{\pi}}{4 s^{5/2}}}$$

- A different approach that lets us practice a powerful

II Formula: #32

$$\boxed{\mathcal{L}\left[\int_0^t f(v)\,dv\right] = \frac{F(s)}{s}, \text{ where } F(s) = \mathcal{L}[f(t)]}$$

Note that $t^{3/2} = \frac{3}{2}\int_0^t t^{1/2}\,dt \quad \left(= \frac{3}{2}\frac{t^{3/2}}{3/2} = t^{3/2} \right)$

Then

$$\mathcal{L}[t^{3/2}] = \mathcal{L}\left[\frac{3}{2}\int_0^t t^{1/2}\,dt\right]$$

$$= \frac{3}{2}\mathcal{L}\left[\int_0^t t^{1/2}\,dt\right]$$

$f(t) = t^{1/2} = \sqrt{t}$

$F(s) = \mathcal{L}[\sqrt{t}] \overset{\#5}{=} \frac{\sqrt{\pi}}{2s^{3/2}}$

$$= \frac{3}{2}\left(\frac{\sqrt{\pi}/2s^{3/2}}{s}\right)$$

$$= \frac{3\sqrt{\pi}}{4\, s^{1+3/2}}$$

$$= \boxed{\frac{3\sqrt{\pi}}{4\, s^{5/2}}}$$