6.3 An example of a regular singular point

Consider the Cauchey-Euler ODE { see attached $\boxed{\text{4.7 notes}}$ }

$$2x^2 y'' - xy' + (1+x)y = 0$$

- So here we have a (regular) singular point @ $x=0$ since $2x^2$ vanishes @ $x=0$ and the y'' term vanishes so... there would only be one solu'n to the first order ODE remaining near $x=0$ - things collapse.

{ see project 5 : Cauchey-Euler Problems : let $y = x^r$ }
↳ we got a new ODE w/o a sing. point

- Here we will let

$$y = x^r(a_0 + a_1 x + \cdots + a_n x^n + \cdots)$$

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^{n+r}}$$

then $\boxed{y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}}$

and $\boxed{y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}}$

- Insert into the ODE:

$$2x^2 y'' - x^1 y' + (1+x^1) y = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2\, a_n (r+n)(r+n-1) x^{r+n-2+2}$$

$$- \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1+1} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

can't reset to $n=2$ since x^r prevents vanishing

Clean up exponents

$$\sum_{n=0}^{\infty} 2\, a_n (r+n)(r+n-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (r+n) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \boxed{\sum_{n=0} a_n x^{n+r+1}} = 0$$

$\leftarrow m$

let $\begin{cases} m = n+1 \\ n = m-1 \end{cases}$

- re-index to match powers:

$$\Rightarrow \sum_{n=0}^{\infty} 2\, a_n (r+n)(r+n-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (r+n) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \boxed{\sum_{m=1}^{\infty} a_{m-1} x^{m+r}} = 0$$

- expand out the 1^{st} sums to match starting pts.

$$2 \times a_0 (r+0)(r+0-1) x^{0+r} - a_0 (r+0) x^{r+0} + a_0 x^{r+0}$$

$$+ \sum_{n=1}^{\infty} \left[2\, a_n (r+n)(r+n-1) - a_n (r+n) + a_n + a_{n-1} \right] x^{n+r} = 0$$

- clean up {eliminate 0}, collect a_n and factor out:

$$\Rightarrow \boxed{a_0 \left[2r(r-1) - r + 1 \right] x^r + \sum_{n=1}^{\infty} \left\{ \left[2(r+n)(r+n-1) - (r+n) + 1 \right] a_n + a_{n-1} \right\} x^{r+n} = 0}$$

- Linear Independent power terms:

x^r: $2r(r-1) - r + 1 = 0 \Rightarrow 2r^2 - 2r - r + 1 = 0$

$\Rightarrow 2r^2 - 3r + 1 = 0$ factors $(r-1)(2r-1) = 0$

"indicial eqn"

we see $\boxed{r_1 = 1,\ r_2 = \frac{1}{2}}$ "exponents at the singularity"

x^{r+n}: see next page

x^{r+n}: $\{\ \} = 0$ so

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0$$

• Solve for a_n {the highest index}

$$a_n = -\frac{a_{n-1}}{(r+n)(2(r+n-1) - 1) + 1}$$

$$(r+n)(2(r+n)-3)+1$$
$$= z(2z-3)+1$$
$$= 2z^2-3z+1$$
$$(2z-1)(z-1)$$

$$\boxed{a_n = -\frac{a_{n-1}}{[2(r+n)-1][(r+n)-1]}}$$

Next we know $r = 1$ and $\frac{1}{2}$ so apply independently ...

• Insert $r_1 = 1$

$$a_n = -\frac{a_{n-1}}{[2(1+n)-1][\cancel{1}+n-\cancel{1}]}$$

$$\boxed{a_n = -\frac{a_{n-1}}{(2n+1)n}}$$ for $r_1 = 1$

Recurssion Relation for $r_1 = 1$

• expand out a few terms ...

$$a_1 = -\frac{a_0}{3\cdot 1}$$

$$a_2 = -\frac{a_1}{5\cdot 2} = -\frac{1}{5\cdot 2}\left(-\frac{a_0}{3\cdot 1}\right) = +\frac{a_0}{(3\cdot 5)\cdot(1\cdot 2)}$$

$$a_3 = -\frac{a_2}{7\cdot 3} = -\frac{a_0}{(3\cdot 5\cdot 7)(1\cdot 2\cdot 3)}$$

general $$\boxed{a_n = \frac{(-1)^n}{[3\cdot 5\cdot 7\cdots(2n+1)]\cdot n!}}$$ for $r_1 = 1$

Recurssion for r_1

- now insert $r_2 = \frac{1}{2}$

$$a_n = -\frac{a_{n-1}}{\left[2\left(\frac{1}{2}+n\right)-1\right]\left[\left(\frac{1}{2}+n\right)-1\right]}$$

$$a_n = -\frac{a_{n-1}}{[\cancel{2}n+\cancel{1}-\cancel{1}]\,\underbrace{[1+2n-2]}_{2n-1}\,\cancel{\tfrac{1}{2}}}$$

$$\boxed{a_n = -\frac{a_{n-1}}{n(2n-1)}}$$ for $r_2 = 1/2$
recursion for $r_2 = 1/2$

- expand out some terms

$$a_1 = -\frac{a_0}{1\cdot 1}$$

$$a_2 = -\frac{a_1}{2\cdot 3} = +\frac{a_0}{(1\cdot 2)(1\cdot 3)}$$

$$a_3 = -\frac{a_2}{3\cdot 5} = -\frac{a_0}{(1\cdot 2\cdot 3)(1\cdot 3\cdot 5)}$$

$\vdots$

general $$\boxed{a_n = \frac{(-1)^n}{n!\,(1\cdot 3\cdot 5\cdots(2n-1))}\,a_0}$$ for $r_2 = \frac{1}{2}$
recursion for $1/2$

- Simplify: mult. top & bottom by $2\cdot 4\cdot 6\cdots 2n = 2^n n!$

$$\boxed{a_n = \frac{(-1)^n 2^n}{(2n+1)!}\,a_0} \qquad \boxed{a_n = \frac{(-1)^n 2^n}{(2n)!}\,a_0}$$

$r_1 = 1$ $\qquad$ $r_2 = 1/2$

Q: is not a_n for $r_2 = 1/2 \Rightarrow \frac{(-1)^n 2^n}{2n-1\,!}$

Now we re-insert a_n into the series: $r=1$

$\boxed{r_1=1}$ $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+(1)}$ put x^1 out front

$$y_1(x) = a_0 x^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right]$$

$\boxed{r_2=1/2}$

$$y_2(x) = a_0 x^{1/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right]$$

- general solution to $\boxed{2x^2y'' - xy' + (1+x)y = 0}$

is $y_{gen}(x) = c_1 x^1 \left[\sum_{0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right] + c_2 x^{1/2} \left[\sum_{0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right]$

- use I.C. to determine c_1 & c_2

- Now we should examine convergence:

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ < 1 conv. > 1 div $= 1$ inconclusive

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

$\boxed{r_1=1}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(2(n+1)+1)!} x^{n+1}}{\frac{2^n}{(2n+1)!} x^n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(2n+1)!}{(2n+3)!} |x^{n+1-n}|$$

$$= \lim_{n\to\infty} \frac{2\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} |x|$$

$$= \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = \boxed{0} \cdot |x| \quad \text{so} \quad \boxed{x \in (-\infty, \infty)}$$

$\boxed{r_2=1/2}$ same results will be discovered

Now we re-insert a_n into the series: $r=1$

$\boxed{r_1=1}$ $$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$ put x^1 out front

$$y_1(x) = a_0 x^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right]$$

$\boxed{r_2=\tfrac{1}{2}}$ $$y_2(x) = a_0 x^{1/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right]$$

- general solution to $\boxed{2x^2y'' - xy' + (1+x)y = 0}$ is

$$y_{gen}(x) = c_1 x^1\left[\sum_{0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right] + c_2 x^{1/2}\left[\sum_{0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right]$$

- use I.C. to determine c_1 & c_2

- Now we should examine convergence:

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ < 1 conv. / > 1 div / $= 1$ inconclusive

$$\frac{a/b}{c/d} = \frac{ad}{bc}$$

$\boxed{r_1=1}$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(2(n+1)+1)!}x^{n+1}}{\frac{2^n}{(2n+1)!}x^n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n}\cdot\frac{(2n+1)!}{(2n+3)!}\left|x^{n+1-n}\right|$$

$$= \lim_{n\to\infty} \frac{2\,\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}|x|$$

$$= \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = \boxed{0}\cdot|x| \quad \text{so} \quad \boxed{x\in(-\infty,\infty)}$$

$\boxed{r_2=\tfrac{1}{2}}$ same results will be discovered

4.7 Cauchey-Euler {Project 5}

Consider $a x^{\boxed{2}} y^{\boxed{''}} + b x^{\boxed{1}} y^{\boxed{'}} + c x^{0} y^{(0)} = 0$ *

• we have an issue @ $x = 0$

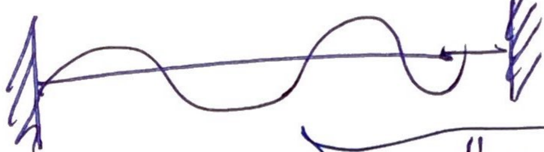
$$\Rightarrow \div a x^2 \quad y'' + \frac{bx}{ax^2} y' + \frac{c}{ax^2} y = 0$$

• Called a singular point:

Bessle funct

• Applications

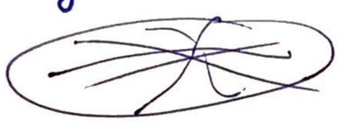
1-D:



$y'' + y = 0$

sin & cos

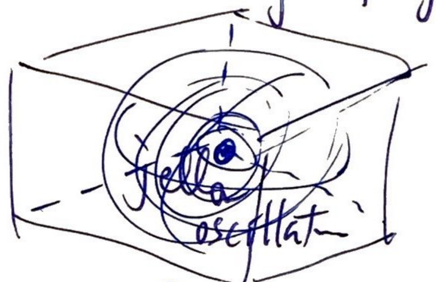
2-D radial only:



Drum head

$r y'' + y = 0$

3-D radial only:



[spherical source

Solid

$r^2 y'' + y = 0$

→ legenre poly.

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* $\sum_{n=0}^{N} a_n x^n y^{(n)} = 0$ for N^{th} order ODE

* To solve let $y(x) = x^r$ then $y' = rx^{r-1}$ & $y'' = r(r-1)x^{r-2}$

the ODE: $ax^2y'' + bxy' + cy = 0$ becomes

$\Rightarrow ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r = 0$

$\Rightarrow \underbrace{[ar(r-1) + b \cdot r + c]}_{=0}\, x^r = 0$

L. Indep. $=0$ since $x^r \neq 0$

"Like a characteristic eqn":

"auxillary" eqn $\longrightarrow$ $\boxed{ar(r-1) + br + c = 0}$ also called the Indicial Eqn.

- three results: $\begin{cases} \text{I. real distinct} \\ \text{II. real double} \\ \text{III. imaginary} \end{cases}$

$\boxed{\text{For } x>0}$

$\boxed{\text{I}}$ if $y = x^r$ with $\boxed{r_1 \neq r_2}$ then the gen soln of the ODE is just

$$\boxed{y(x) = C_1 x^{r_1} + C_2 x^{r_2}}$$

$x>0$

use I.C. to get c_1 & c_2

$\boxed{\text{II}}$ Double ROOTS $\boxed{r_1 = r_1}$

{Recall for Const. coeff. we use $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$}

· Here it can be shown that, via reduction of order

$$\boxed{y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln(x)}$$

$x>0$

III Complex Roots

$r_{1,2} = \lambda \pm \mu i$

real, img.

Use Trick: $\boxed{a^b = e^{b\ln(a)}}$

So then $x^{\lambda \pm \mu i} = e^{(\lambda \pm \mu i)\ln x}$

$= e^{\lambda \ln x} \, e^{\pm \mu \ln x \, i}$ Euler formula

$= x^{\lambda}\left[\cos(\mu \ln x) + i \sin(\mu \ln x)\right]$

- We use both the real & img parts to get the Gen. soln like we did w/ const. coefficients:

$$\boxed{y(x) = c_1 x^{\lambda} \cos(\mu \ln(x)) + c_2 x^{\lambda} \sin(\mu \ln(x))}$$

$x > 0$

For $x < 0$: we let $\eta = -x$, then $\eta > 0$

so define $u(\eta) \equiv y(-\eta)$

then $u'(\eta) = -y'(x)$ & $u''(\eta) = y''(x)$

the ODE becomes

$$\boxed{a(-\eta)^2 u'' + b(-\eta)(-u') + cu = 0}$$

now let $x = -\eta \Rightarrow$

auxillary eqn is $ar(r-1)+br+c=0$,

case $\boxed{\text{I}}$ $u(\eta)=c_1\eta^{r_1}+c_2\eta^{r_2}$

but this is $\boxed{y(x)=c_1(-x)^{r_1}+c_2(-x)^{r_2}}$ $x<0$

real & distinct roots.

Compare to $x>0$

$\boxed{y(x)=c_1x^{r_1}+c_2x^{r_2}}$

Combine to get $\boxed{y(x)=c_1|x|^{r_1}+c_2|x|^{r_2} \quad x\neq 0}$

- Likewise for $\boxed{\text{II}}$ & $\boxed{\text{III}}$:

$$\boxed{y(x)=c_1|x|^{r}+c_2|x|^{r}\ln|x| \qquad x\neq 0}$$

III $\rightarrow$ $$\boxed{y(x)=c_1|x|^{\lambda}\cos(\mu\ln|x|)+c_2|x|^{\lambda}\sin(\mu\ln|x|) \quad x\neq 0}$$