Chapter 6 | Series Solution Method

- When an analytical method will not seem possible, we can try an ∞-series soln.

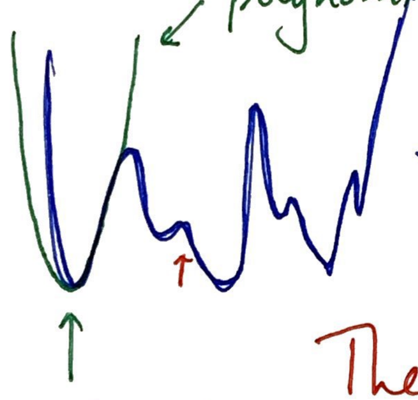
-OR- when we want to provide some one with an analytical polynomial approximation we will find the ∞-series method helpful.

- Recall in Calc II we introduced the Taylor Series and also the power series as approximations to functions.

EX

polynomial approximation about $x = x_0$

messy function



$\sqrt{\cos^2(t) \cdot t^t}$ etc.

$x = x_0$

The approximation only is valid near $x = x_0$.

- In using ∞-series in ODE's we face the same dilema - We must stay close to the point of expansion.

$\boxed{6.1}$ is a review of Calc II ∞-series expansions.

$\boxed{6.2}$ Using ∞ Series to Solve ODE's

Consider $p(x)y'' + q(x)y' + r(x)y = 0$

If $\dfrac{q(x)}{p(x)}$ and $\dfrac{r(x)}{p(x)}$ are capable of being replaced by a Taylor series about some point x_0 we call q/p and r/p analytic and the point x_0 is called an ordinary point of the ODE.

If x_0 is not analytic it is a singular point.

The idea in ∞-series solutions to ODE's is to assume the solution can be expanded in an ∞-series.

- Form: $\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n}$

expanded: $y = a_0 (x-x_0)^0 + a_1 (x-x_0)^1 + a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots$

- Diff't

$$y' = 0 + \overset{n=1}{a_1} + \overset{n=2}{2a_2 (x-x_0)} + \overset{n=3}{3a_3 (x-x_0)^2} + \cdots$$

$$\Rightarrow \quad \boxed{y' = \sum_{n=1}^{\infty} a_n n (x-x_0)^{n-1}}$$

$$y' = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x_0)^3 + \cdots$$

and diff't again $\Rightarrow$ $$y'' = 0 + \overset{n=2}{2a_2\cdot 1} + \overset{n=3}{3\cdot 2a_3(x-x_0)^1} + \overset{n=4}{4\cdot 3a_4(x-x_0)^2} + \cdots$$

$$\boxed{y'' = \sum_{n=2}^{\infty} n(n-1)(x-x_0)^{n-2}}$$

- Insert these into the ODE
- Simplify and establish a <u>recursion relationship</u> particular to the ODE.

P R O C E S S

(i) $y = \sum a_n (x-x_0)^n$ $\longrightarrow$ (ii) $\boxed{\text{ODE}_{gn}}$ $\longrightarrow$ $a_{n+1} = f(a_n)$

$\boxed{\text{form}}$ $\qquad$ $\boxed{\text{recursion relationship}}$

$y =$ particular series solution $\longleftarrow$ $y = \sum a_n (x-x_0)^n$ $\longleftarrow$

$\boxed{\text{insert } a_n}$

Ex Solve $y''+y=0$ via a series sol'n expanded about $x_0=0$

I Find the Recursion Relationship

(i) form. $y = \sum_{n=0}^{\infty} a_n x^n$

(ii) derivatives

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

(iii) Into ODE

$$y'' + y = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

(iv) group like powers: re-index then align series start

m

let $m=n-2$ then $n=m+2$

$$\underbrace{\sum_{m=0}^{\infty} a_{m+2} (m+2)(m+2-1) x^m}_{\text{let } m \text{ be replaced by } n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

- consolidate series

$$\boxed{\sum_{n=0}^{\infty} [a_{n+2} (n+2)(n+1) + a_n] x^n = 0}$$

- all coeffients, [ ], of a powers series set eq to zero must also be zero due to this Linear In

Since $a_{n+2}(n+2)(n+1) + a_n = 0$

we have our recursion relationship

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)}}$$ particular to $y'' + y = 0$

$\boxed{\text{II}}$ Build the series

(v) expand the recursion relationship

k	n		
1	$n=0$:	$a_{0+2} = -\frac{a_0}{(0+2)(0+1)}$	$= -\frac{1}{2\cdot 1} a_0$
1	$n=1$:	$a_3 = -\frac{a_1}{(1+2)(1+1)}$	$= -\frac{1}{3\cdot 2} a_1$
2	$n=2$:	$a_4 = -\frac{a_2}{(2+2)(2+1)}$	$= -\frac{1}{4\cdot 3} a_2$
		but we know a_2	$a_4 = -\frac{1}{4\cdot 3}\left(-\frac{1}{2\cdot 1}\right) a_0$
2	$n=3$	$a_5 = -\frac{a_3}{(3+2)(3+1)}$	$= \frac{-1}{5\cdot 4}\left(\frac{-1}{3\cdot 2}\right) a_1$
3	$n=4$	$a_6 = -\frac{a_4}{(4+2)(4+1)}$	$= -\frac{1}{6\cdot 5}\left(\frac{1}{4\cdot 3}\cdot\frac{1}{2\cdot 1}\right) a_0$
3	$n=5$	$a_{5+2} = -\frac{a_5}{(5+2)(5+1)}$	$= -\frac{1}{7\cdot 6}\left(\frac{1}{5\cdot 4\cdot 3\cdot 2}\right) a_1$

• generalize terms: if possible

$$a_{odd} = \frac{(-1)^{n}}{(n+2)!} a_1 \quad ?$$

$$a_{even} = \frac{(-1)^{n}}{(n+2)!} a_0$$

odd $= 2k+1$ even $= 2k$

even: $a_{2k} = \dfrac{(-1)^k}{(2k)!} a_0 \quad , k = 0, 1, 2, 3 \ldots$

odd $a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!} a_1 \quad , k = 0, 1, 2 \cdots$

this process often will not be practical!

(vi) build series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

$$y = a_0 + a_1 x + \left(-\frac{1}{2\cdot 1} a_0\right) x^2 + \left(\frac{-1}{3\cdot 2} a_1\right) x^3 + \left(\frac{1}{4\cdot 3\cdot 2\cdot 1} a_0\right) x^4 + \left(\frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1} a_1\right) x^5 + \ldots$$

· factor out a_0 and a_1:

$$y = a_0 - \frac{1}{2!} a_0 x^2 + \frac{1}{4!} a_0 x^4 - \frac{1}{6!} a_0 x^6 + - \ldots + a_1 x^1 - \frac{1}{3!} a_1 x^3 + \frac{1}{5!} a_1 x^5 - \frac{1}{7!} a_1 x^7 + \ldots$$

$$y = \underbrace{a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \ldots\right]}_{\text{even}} + \underbrace{a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \ldots\right]}_{\text{odd}}$$

Recall

$$\cos(x) = \overset{k=0}{1} - \overset{k=1}{\frac{x^2}{2!}} + \overset{k=2}{\frac{x^4}{4!}} - \overset{k=3}{\frac{x^6}{6!}} + \cdots = \boxed{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}}$$

and

$$\sin(x) = \underset{k=0}{x} - \underset{k=1}{\frac{x^3}{3!}} + \underset{k=2}{\frac{x^5}{5!}} - \underset{k=3}{\frac{x^7}{7!}} + = \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}}$$

Then

$$\boxed{y(x) = a_0 \cos(x) + a_1 \sin(x)}$$ an analytic solution!!

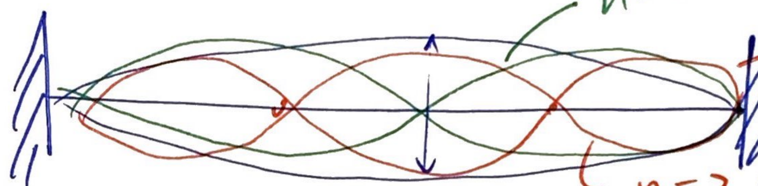
(vii) Apply the I.C. to determine a_0 & a_1

we will find $y(0) = a_0$, $y'(0) = a_1$

- ∞ - series is regularly used with periodic phenomena - oscillations

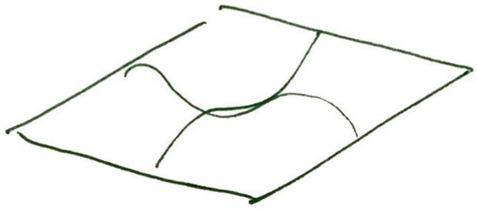
we saw that the cosine and sine functions came out of $y'' + y = 0$ eqn. cartesian (1-D)

1-D:



$$\boxed{\sum \frac{(-1)^{2k}}{(2k)!} x^{2k}}$$

2-D:



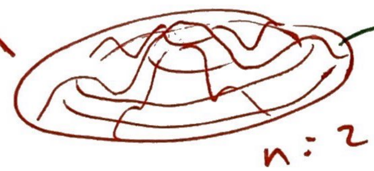
vibrations on a plate

2-D polar:

Drum Head vibration

Bessle Function

$$J_0(\lambda r) = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{(\lambda r/4)^k}{(k!)^2}}$$



what function describes this shape? Bessle!

It is like a sine and cosine but the amplitude decays as one travels away from the center.

EX Solve via ∞-series $y'' - xy = 0$

I Recursion Relationship

(i) form
$$y = \sum_{n=0}^{\infty} a_n x^n$$

(ii) diff't
$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

(iii) ODE
$$\left[\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}\right] - x\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

$\boxed{\begin{array}{l} m = n-2 \\ n = m+2 \end{array}}$

(iv) match powers
$$\left[\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)\, x^m\right] - \left[\sum_{n=0}^{\infty} a_n x^{n+1}\right] = 0$$

$\boxed{\begin{array}{l} m = n+1 \\ n = m-1 \end{array}}$

$$\left[\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)\, x^m\right] - \left[\sum_{m=1}^{\infty} a_{m-1} x^m\right] = 0$$

(v) match starting index
our starting index is not 0 so expand 1st series out one index value:

$$\left[a_{0+2}(0+2)(0+1)x^0 + \sum_{m=1}^{\infty} a_{m+2}(m+2)(m+1)\, x^m\right] - \left[\sum_{m=1}^{\infty} a_{m-1} x^m\right] = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) - a_{n-1}\right] x^n = 0$$

$x^0 : 2a_2 = 0 \longrightarrow \boxed{a_2 = 0}$

$x^1 : a_{n+2}(n+2)(n+1) = a_{n-1}$

or

$$\boxed{\begin{aligned} a_2 &= 0 \\ a_{n+2} &= \frac{a_{n-1}}{(n+2)(n+1)} \end{aligned}}$$

Recursion Rela'n for $y'' - xy = 0$.

II build series

(vi) expand.

$n=1 \qquad a_3 = \dfrac{a_0}{(1+2)(1+1)} = \boxed{\dfrac{a_0}{3\cdot 2}}$

$n=2 \qquad a_4 = \dfrac{a_1}{(2+2)(2+1)} = \boxed{\dfrac{a_1}{4\cdot 3}}$

$a_2 = 0$

$n=3 \qquad a_5 = \dfrac{a_2}{(3+2)(3+1)} = \dfrac{a_2}{5\cdot 4} = 0$

$n=4 \qquad a_6 = \dfrac{a_3}{6\cdot 5} = \boxed{\left(\dfrac{1}{6\cdot 5}\right)\dfrac{a_0}{3\cdot 2}}$

$n=5 \qquad a_7 = \dfrac{a_4}{7\cdot 6} = \boxed{\dfrac{1}{7\cdot 6}\left(\dfrac{a_1}{4\cdot 3}\right)}$

$n=6 \qquad a_8 = \dfrac{a_5}{8\cdot 7} = 0$

$n=7 \qquad a_9 = \dfrac{a_6}{9\cdot 8} = \boxed{\dfrac{1}{9\cdot 8}\left(\dfrac{a_0}{6\cdot 5\cdot 3\cdot 2}\right)}$

$\vdots$

repeats in three's

$\vdots$

- expand the series out a few terms

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \dots$$

$$y = a_0 + a_1 x + 0 + \left(\frac{a_0}{5}\right)x^3 + \left(\frac{a_1}{12}\right)x^4 + 0 + \left(\frac{a_0}{180}\right)x^6 + \left(\frac{a_1}{504}\right)x^7$$

- group by a_0 and a_1

$$\boxed{y(x) \approx a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots\right] + a_1\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \dots\right]}$$

poly-nomial approx. 7^{th} order.

NOTE For $x = 0.1 \rightarrow x^3 = 0.001$

$x^4 = 0.0001$

$x^6 = 0.000001$ ↙ very small.

we are expanding about $x_0 = 0$ so we need to stay close to $x = 0$

NOTE → $y(0) = a_0[1 + 0 + 0 + 0 \dots] + a_1[0 + 0 + 0 \dots]$

so $\boxed{a_0 = y(0)}$ I.C.

likewise → $y'(x) = 0 + a_1 +$ terms with x

$\boxed{a_1 = y'(0)}$ the other I.C.

summary

$$\boxed{y \approx a_0\left[1 + \frac{x^3}{6}\right] + a_1\left[x + \frac{x^4}{12}\right]}$$

good near $x = 0$

let say $y(0) = 1$, $y'(0) = 1$

Ex Solve the previous ODE but produce an approx. polynomial solution that is "good" near $x=-2$: $y''-xy=0$ 12

(i) assume the form $y=\sum_{n=0}^{\infty} a_n (x-(-2))^n = \sum_{n=0}^{\infty} a_n (x+2)^n$

(ii) $y'=\sum_{n=1}^{\infty} a_n n (x+2)^{n-1}$, $y''=\sum_{n=2}^{\infty} a_n n(n-1)(x+2)^{n-2}$

(iii) ODE

$$\left[\sum_{n=2}^{\infty} a_n n(n-1)(x+2)^{n-2}\right] - x\left[\sum_{n=0}^{\infty} a_n (x+2)^n\right] = 0$$

⊛ Modify x : $[(x+2)-2]$

$$\Rightarrow \left[\sum_{n=2}^{\infty} a_n n(n-1)(x+2)^{n-2}\right] - (x+2)\left[\sum_{n=0}^{\infty} a_n (x+2)^n\right] + 2\left[\sum_{n=0}^{\infty} a_n (x+2)^n\right] = 0$$

$$\Rightarrow \left[\sum_{n=2}^{\infty} a_n (n)(n-1)(x+2)^{n-2}\right] - \sum_{n=0}^{\infty} a_n (x+2)^{n+1} + 2\sum_{n=0}^{\infty} a_n (x+2)^n = 0$$

$m=n-2$, $n=m+2$ $\qquad$ $m=n+1$, $n=m-1$ $\qquad$ ok.

$$\left[\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)(x+2)^m\right] - \sum_{m=1}^{\infty} a_{m-1}(x+2)^m + 2\sum_{n=0}^{\infty} a_n (x+2)^n = 0$$

• Now powers match!

But the $\boxed{\text{starting indeces don't match}}$ ... expand out

$$a_2 \cdot 2 \cdot 1 \cdot (x+2)^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}(x+2)^n$$

$$+ a_0 (x+2)^0 + 2\sum_{n=1}^{\infty} a_n (x+2)^n = 0$$

* Factor x^n

$$\boxed{2a_2 + a_0 + \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) - a_{n-1} + 2a_n\right](x+2)^n = 0}$$

match LHS with RHS

x^0 : $2a_2 + a_0 = 0 \longrightarrow \boxed{a_2 = \frac{-1}{2}a_0}$

x^n : $a_{n+2}(n+2)(n+1) = a_{n-1} - 2a_n$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_{n-1} - 2a_n}{(n+2)(n+1)}}$$

now $x_0 = -2$ due to the relocation of the point of expansion.

same as $x_0 = 0$

* <u>expand out</u>

$n=1$ $\qquad a_3 = \dfrac{a_0 - 2a_1}{3 \cdot 2} = \boxed{\left(\dfrac{a_0}{6} - \dfrac{a_1}{3}\right) = a_3}$

$n=2$ $\qquad a_4 = \dfrac{a_1 - 2a_2}{4 \cdot 3} = \dfrac{a_1 - 2\cancel{(}\cancel{+}\frac{1}{2}a_0)}{12}$

$$\boxed{a_4 = \frac{a_1 + a_0}{12}}$$

$n=3$ $\qquad a_5 = \dfrac{a_2 - 2a_3}{5 \cdot 4} = \dfrac{\left(-\frac{1}{2}a_0\right) - 2\left(\frac{a_0}{6} - \frac{a_1}{3}\right)}{20}$

$$a_5 = \frac{-a_0/2 - \frac{2a_0}{6} + \frac{2a_1}{3}}{20}$$

$$a_5 = \frac{-5/6\, a_0 + \frac{2a_1}{3}}{20} = \frac{-5a_0}{120} + \frac{2a_1}{60}$$

* **Build Series**

$$y = a_0 + a_1(x+2) + a_2(x+2)^2 + a_3(x+2)^3 + a_4(x+2)^4 + \cdots$$

$$y = a_0 + a_1(x+2) + \left(-\frac{a_0}{2}\right)(x+2)^2 + \left(\frac{a_0}{6} - \frac{a_1}{3}\right)(x+2)^3 + \left(\frac{a_1 + a_0}{12}\right)(x+2)^4 + \cdots$$

• Seperate terms into a_0 & a_1 : { do we need to ??? }

$$y = a_0\left[1 - \frac{(x+2)^2}{2} + \cdots\right] + a_1\left[(x+2) + \cdots\right]$$

$y' = a_1$ @ $x = 2$ We see that $\begin{cases} a_0 = y(0) \\ a_1 = y'(0) \end{cases}$

let $y(2) = -1$, $y'(2) = 1$ ⇐ I.C.

$\uparrow a_0$ $\uparrow a_1$

$$y = -1 + 1\cdot(x+2) - \left(\frac{-1}{2}\right)(x+2)^2 + \left(\frac{-1}{6} - \frac{1}{3}\right)(x+2)^3 + \left(\frac{1 + (-1)}{12}\right)(x+2)^4$$

$$y \simeq -1 + (x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{2}(x+2)^3 + 0(x+2)^4 + \cdots$$

this polynomial approximates soln near $x = -2$

$$\boxed{y \approx -1 + (x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{2}(x+2)^3}$$

good near $x = -2$ only.

a polynomial approx soln of $y'' - xy = 0$ near $x = -2$

$\boxed{\text{EX}}$ Solve $(1+x^2)y'' + (2-x)y' + 3y = 0$ @ $x=0$

$$\Rightarrow \boxed{y'' + x^2 y'' + 2y' - xy' + 3y = 0}$$

(i) (ii) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

(iii) $\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1) x^{\overbrace{n-2+2}^{n}}$

$+ \sum_{n=1}^{\infty} 2a_n n x^{n-1} - \sum_{n=1}^{\infty} a_n n x^{\overbrace{n-1+1}^{n}} + \sum_{n=0}^{\infty} 3a_n x^n = 0$

$m = n-2$
$n = m+2$

$m = n-1$
$n = m+1$

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{m=0}^{\infty} 2a_{m+1}(m+1)x^m$$

$$- \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

• expand starting index:

$$a_2 \cdot 2 \cdot 1 + a_3 \cdot 3 \cdot 2 x^1 + \sum_{m=2}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=2}^{\infty} a_n n(n-1)x^n$$

$$+ 2a_1 \cdot 1 + 2a_2 \cdot 2 \cdot x^1 + \sum_{m=2}^{\infty} 2a_{m+1}(m+1)x^m$$

$$- a_1 \cdot 1 \cdot x - \sum_{n=2}^{\infty} a_n n x^n + 3a_0 + 3a_1 x + \sum_{n=2}^{\infty} 3a_n x^n$$

$$x^0(2a_2 + 2a_1 + 3a_0) + x^1(6a_3 + 4a_2 - a_1 + 3a_1)$$

$$+ \sum_{n=2}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n n(n-1) + 2a_{n+1}(n+1) - a_n n + 3a_n\right] x^n = 0$$

x^0 : $\boxed{2a_2 + 2a_1 + 3a_0 = 0}$

x^1 : $6a_3 + 4a_2 - a_1 + 3a_1 = 0 \rightarrow 6a_3 + 4a_2 + 2a_1 = 0 \quad \div 2$

$$\boxed{3a_3 + 2a_2 + a_1 = 0}$$

x^n : $(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + [n(n-1) - n + 3]a_n = 0$

$n^2 - n - n + 3$

$\boxed{n^2 - 2n + 3}$

$\cancel{(n-3)(n+1)}$? prime

Recursion Relationship

$$\boxed{\begin{aligned} a_2 &= -a_1 - \tfrac{3}{2}a_0 \\ a_3 &= -\frac{a_1}{3} - \frac{2}{3}a_2 \\ a_{n+2} &= \frac{-2(n+1)a_{n+1} - (n^2-2n+3)a_n}{(n+2)(n+1)} \end{aligned}}$$

Q: Can the bottom relation encompass the individuals?

$$n=0 \quad \begin{cases} a_{0+2} = \dfrac{-2(0+1)a_{0+1} - (0^2 - 2\cdot 0 + 3)a_0}{(0+2)(0+1)} \\ a_2 = \dfrac{-2a_1 - 3a_0}{2} = -a_1 - \dfrac{3}{2}a_0 \quad \checkmark \end{cases}$$

$$n=1 \quad \begin{cases} a_{1+2} = \dfrac{-2(1+1)a_{1+1} - (1^2 - 2\cdot 1 + 3)a_1}{(1+2)(1+1)} \end{cases}$$

$$= \frac{-4a_2 - 2a_1}{3\cdot 2} = \frac{-2a_2 - a_1}{3} = -\frac{a_1}{3} - \frac{2}{3}a_2$$

So we can ignore the top two relations:

$$\boxed{a_{n+2} = \frac{-2(n+1)a_{n+1} - (n^2-2n+3)a_n}{(n+2)(n+1)}}$$

Build Series ...

$$n=0: \quad a_2 = \frac{-2(0+1)a_{0+1} - (0^2-2\cdot 0+3)a_0}{(0+2)(0+1)} = -a_1 - \frac{3}{2}a_0$$

$$n=1: \quad a_3 = \frac{-2(1+1)a_{1+1} - (1^2-2\cdot 1+3)a_1}{(1+2)(1+1)} = -\frac{a_1}{3} - \frac{2}{3}a_2$$

$$n=2: \quad a_4 = \frac{-2(2+1)a_{2+1} - (2^2-2\cdot 2+3)a_2}{(2+2)(2+1)}$$

$$a_4 = \frac{-4a_3 - 3a_2}{4\cdot 3} = -\frac{a_3}{3} - \frac{a_2}{4}$$

$$n=3: \quad a_{3+2} = \frac{-2(3+1)a_{3+1} - (3^2-2\cdot 3+3)a_3}{(3+2)(3+1)}$$

$$= \frac{-8a_4 - 6a_3}{5\cdot 4} = -\frac{2}{5}a_4 - \frac{3}{10}a_3$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$\boxed{\text{4th order}}$

$$y \approx a_0 + a_1 x + \left(-a_1 - \frac{3}{2}a_0\right)x^2 + \left(-\frac{a_1}{3} - \frac{2}{3}a_2\right)x^3 + \left(-\frac{a_3}{3} - \frac{a_2}{4}\right)x^4 \quad \text{STOP}$$

$$\left[-a_1 - \frac{3}{2}a_0\right]$$

$$-\frac{a_1}{3} - \frac{2}{3}a_2$$

$$= \left[-\frac{a_1}{3} - \frac{2}{3}\left(-a_1 - \frac{3}{2}a_0\right)\right]$$

we could continue to insert a_3, a_4 ... relations in terms of a_0 & a_1 but if we know the I. Conds, assign a_0 & a_1 1st then build series ...

- Let $y(0)=1$ and $y'(0)=-1$
then $\boxed{a_0 = 1 \quad \text{and} \quad a_1 = -1}$ Seeds

- $a_2 = -a_1 - \frac{3}{2}a_0 = -(-1) - \frac{3}{2}(1) = \boxed{-1/2 = a_2}$

- $a_3 = -\frac{a_1}{3} - \frac{2}{3}a_2 = -\frac{(-1)}{3} - \frac{2}{3}\left(-\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3} = a_3}$

- $a_4 = -\frac{a_3}{3} - \frac{a_2}{4} = -\frac{(2/3)}{3} - \frac{(-1/2)}{4} = -\frac{2}{9} + \frac{1}{8} = \frac{-16+9}{72}$

$$\boxed{a_4 = -7/72}$$

Then

$$y \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$= 1 + (-1)x + \left(-\frac{1}{2}\right)x^2 + \left(\frac{2}{3}\right)x^3 + \left(-\frac{7}{72}\right)x^4$$

$$\boxed{y \simeq 1 - x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{7}{72}x^4}$$ Boss's form

Must stay near $x=0$

desmos says good to ± 0.7 ??
need x^5 term to determine