(4.5 is to be skipped) Do NOT use the "D" operator in this class

$\boxed{4.6}$ Variation of Parameters

- Recall in 4.4 we examined Undetermined coefficients method. That method is limited to simple driving function like $e^{\alpha t}$, $\sin\beta t$, $\cos\beta t$, t^n or any combinations of these.
- For more complicated driving functions we consider "variations of parameters" { or Laplace Transform, or Numerical Methods, or ∞-series
- Recall $W(f_1, f_2, \dots, f_n)$ is used to determine Linear Dependency

$\boxed{Ex}$ If f & g are Lin Indep. then

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \neq 0$$

then $\{f, g\}$ is a fund. solution set.

$\boxed{Ex}$ $f = 9\cos(2x)$, $g = 2\cos^2 x - 2\sin^2 x$ — same function

$f' = -18\sin(2x)$, $g' = -4\cos x \sin x - 4\sin x \cos x$

$\frac{\sin 2x}{2}$ $\qquad$ $\frac{\sin 2x}{2}$

$$W = \begin{vmatrix} 9\cos 2x & 2\cos^2 x - 2\sin^2 x \\ -18\sin 2x & -2\sin 2x - 2\sin 2x \end{vmatrix}$$

$\sin 2x = 2\sin x\cos x$

$= (9\cos 2x)(-4\sin 2x) - (-18\sin 2x)(2)(\cos^2 x - \sin^2 x)$ $\quad 2\sin(2x)?$

$= -36 \underbrace{\sin(4x)}_{2} + 18\sin(2x)\sin(2x)$

$=$? b/c $= \cos 2x$

ow we outline the Var. of Parameter processess.

- Consider the ODE:

$$\boxed{y'' + q(t)\,y' + r(t)\,y = g(t)}$$

- assume $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ is <u>known</u>

- assume $Y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$

  where u_1, u_2 are to be determined

- diff't: $Y_p'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$

$$= u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

- Require $\boxed{u_1' y_1 + u_2' y_2 = 0}$ a condition on u_1 & u_2

  then $Y_p'(t) = u_1 y_1' + u_2 y_2'$

- Diff't again:

$$Y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

- ODE: $[u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'']$

$$+ q \cdot [u_1 y_1' + u_2 y_2'] + r \cdot [u_1 y_1 + u_2 y_2] = g(t)$$

- factor out u_1 & u_2

$$(u_1'y_1' + u_2'y_2') + u_1\big[\underbrace{y_1'' + qy_1' + ry_1}_{=0}\big] + u_2\big[\underbrace{y_2'' + qy_2' + ry_2}_{=0}\big] = g$$

- But y_1 & y_2 solve the homogeneous ODE so the [ ] terms are 0

- $\Rightarrow$ $\boxed{\begin{aligned} u_1'y_1' + u_2'y_2' &= g(t) \\ u_1'y_1 + u_2'y_2 &= 0 \end{aligned}}$ Two eqns & two unknowns

Solve for u_1' & u_2', then integrate to get $u_1(t)$ and $u_2(t)$ then build $Y_p = u_1y_1 + u_2y_2$

- Solve using Cramer's Rule:

$$\begin{aligned} ax + by &= c \\ dx + fy &= h \end{aligned} \qquad x = \frac{\begin{vmatrix} c & b \\ h & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & f \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & c \\ d & h \end{vmatrix}}{\begin{vmatrix} a & b \\ d & f \end{vmatrix}}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{W}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{W}, \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

-OR-

$$u_1' = -\frac{g\,y_2}{W}\ ; \quad u_2' = \frac{g\,y_1}{W}, \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Integrals

$$u_1 = -\int \frac{g(t)\,y_2(t)}{y_1y_2' - y_1'y_2}\,dt, \quad u_2 = \int \frac{g(t)\,y_1(t)}{y_1y_2' - y_1'y_2}\,dt$$

• piece Y_p together: $Y_p = u_1 y_1 + u_2 y_2$

$$\boxed{Y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)}\,dt}$$

• gen soln of $y'' + qy' + ry = g$ is

$$\boxed{y_{gen}(t) = c_1 y_1 + c_2 y_2 - y_1\int \frac{y_2 g}{W}\,dt + y_2\int \frac{y_1 g}{W}\,dt}$$

where y_1 & y_2 are solutions to $y'' + qy' + ry = 0$

"Canned solution"

Review methods for non-homogeneous:

1. Undet. Coeff if q & r = const. g = easy
2. Variation of Parameters if the integrals are tractable

future methods

3. Laplace transforms if g is piece-wise
4. ∞-series solutions otherwise
5. Numerical Methods

$\boxed{\text{Ex}}$ Solve $2y'' + 18y = 6\tan(3t)$ via V. of Params. Formula

$$y'' + 9y = 3\tan(3t)$$

- y_c: $r^2 + 3^2 = 0 \rightarrow y_c = c_1 \underbrace{\cos(3t)}_{y_1} + c_2 \underbrace{\sin(3t)}_{y_2}$

- $W := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = \boxed{3}$

- $Y_p = -y_1 \cdot \int \frac{y_2 g}{W}\,dt + y_2 \cdot \int \frac{y_1 g}{W}\,dt$

populate formula

$$Y_p = \overset{y_1}{\cos(3t)} \overset{u_1}{\left(-\int \frac{\sin(3t)\,6\tan(3t)}{3}\,dt\right)} + \overset{y_2}{\sin(3t)} \overset{u_2}{\int \frac{\cos(3t)\,6\tan(3t)}{3}\,dt}$$

$$= -\cos(3t)(2)\int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)(2)\underbrace{\int \sin(3t)\,dt}_{-\frac{1}{3}\cos(3t)}$$

$\sin^2(t) = 1 - \cos^2(3t)$

$$= -2\cos(3t)\int\left(\frac{1}{\cos(3t)} - \frac{\cos^2(3t)}{\cos(3t)}\right)dt - \frac{2}{3}\sin(3t)\cos(3t)$$

$$= -2\cos(3t)\left[\int \sec(3t)\,dt - \cancel{\frac{\sin(3t)}{3}}\right] \cancel{- \frac{2}{3}\sin(3t)\cos(3t)}$$

$$= -\frac{2\cos(3t)}{3}\left[\ln|\sec(3t) + \tan(3t)|\right]$$

- $\boxed{y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{2}{3}\cos(3t)\ln|\sec(3t) + \tan(3t)| + c}$

end $\underline{4.6}$