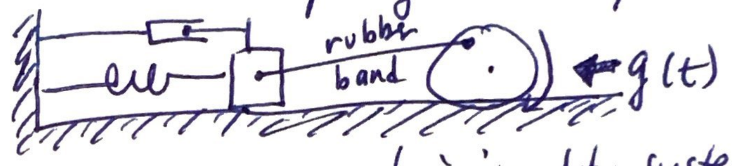
$\boxed{4.4}$ Non-homogeneous 2nd order ODE's.

- Homogeneous: $y'' + p(t)y' + q(t)y = 0$
- non-homog: $\boxed{y'' + p(t)y' + q(t)y = g(t)}$

We frequently call $g(t)$ the <u>driving function</u>

{ the term comes from a spring-mass system w/ a driver hooked to it



driving the system }

THM: If $Y_p(t)$ is a solution to the non-hom, ODE then the general solution is

$$\boxed{y(t) = y_c(t) + Y_p(t)}$$

particular solution

where $y_c(t)$ is the gen. solution to the homogeneous ODE, and for second order has the form $y_c = c_1 y_1 + c_2 y_2$

complementary soln

Proof: $y' = y_c' + Y_p'$, $y'' = y_c'' + Y_p''$

Insert into Non-Homog. ODE:

$$(y_c'' + Y_p'') + p(t)(y_c' + Y_p') + q(t)(y_c + Y_p) = g(t)$$

Homog. ODE

$$\underbrace{(y_c'' + p y_c' + q y_c)}_{=0} + \underbrace{(Y_p'' + p Y_p' + q Y_p)}_{\text{non-homogeneous}} = g(t)$$

We can solve non-homog. linear ode using one of two methods: (a) Undetermined Coefficients.
(b) Variation of Parameters

They both have their pro's and con's.

* (a) Undetermined Coefficients.

advantages:
- problem becomes an algebra prob
- complementary soln often is not required.

disadvantages:
- works for a limited $g(t)$
- generally usefull for const. coefficients only.
- must know starting format.

I The Basics

- General Process:
  (i) solve the complementary prob. first.
  (ii) make an educated guess as to Y_p's form.
  (iii) perform derivatives, insert into ODE and
  (iv) solve for the coefficients by matching
  (v) present the $y_{gen} = y_c + Y_p$

- In part III of this section we modify step (ii)
- Part II is sec. 4.5,

$\boxed{\text{Ex}}$ Introductory example. (Basic Steps)

Consider $y'' - 4y' - 12y = 3e^{5t}$, Solve it.

(i) complimentary soln : $\begin{cases} y'' - 4y' - 12y = 0 \\ r^2 - 4r - 12 = 0 \quad r = -2, 6 \end{cases}$

$$y_c = c_1 e^{-2t} + c_2 e^{6t}$$

(ii) Guess $Y_p(t)$ has the form $Y_p(t) = A e^{5t}$

$\{ e^{5t}$ because $g(t) = 3e^{5t} \}$

(iii) • Diff't : $Y_p' = A5e^{5t}$, $Y_p'' = A25e^{5t}$

• Insert into ODE : $(A25e^{5t}) - 4(A5e^{5t}) - 12(Ae^{5t}) = 3e^{5t}$

(iv) Solve for coefficients:

$\div e^{5t}$ $\quad 25A - 20A - 12A = 3$ $\quad \Rightarrow A = -\frac{3}{7}$

$-7A = 3$

(V) Present the gen. Soln:

Particular Soln: $\boxed{Y_p = -\frac{3}{7}e^{5t}}$

Gen. Soln: $\boxed{y(t) = c_1 e^{-2t} + c_2 e^{6t} - \frac{3}{7}e^{5t}}$

↑ ↑

To be determined by I.C.

$y(t_0)$ & $y'(t_0)$.

WARNING: Do not apply the Initial Conditions to only y_c. You must include Y_p

EX Find the gen. sol'n for

$$y'' - 4y' - 12y = \sin(2t)$$

(i) complimentary: done in the previous example $y_c = c_1 e^{-2t} + c_2 e^{6t}$

(ii) Form: $Y_p(t) = A\sin(2t)$

(iii) Diff't and put into ODE :

Y_p'' Y_p' Y_p

$$(-4A\sin(2t)) - 4(2A\cos(2t)) - 12(A\sin(2t)) = \sin(2t)$$

(iv) Solve for coeff's.

$$16A\sin(2t) - 8A\cos(2t) = \sin(2t)$$

match : sin : $16A = 1 \rightarrow A = -1/16$
cos : $-16A = 0 \rightarrow A = 0$ } huh?

- We need to expand our form of Y_p to include not just $\sin(2t)$ but also $\cos(2t)$

I.E. let $\boxed{Y_p = A\cos(2t) + B\sin(2t)}$

- Start again...

(iii) Y_p'' Y_p'

$$(-4A\cos(2t) - 4B\sin(2t)) - 4(-2A\sin(2t) + 2B\cos(2t)) - 12(A\cos(2t) + B\sin(2t)) = \sin(2t)$$

gather like terms:

$$(-16A - 8B)\cos(2t) + (8A - 16B)\sin(2t) = \sin(2t)$$

(iv) match : $\cos(2t)$: $-16A - 8B = 0$
$\sin(2t)$: $8A - 16B = 1$ } $\underline{\underline{A = \frac{1}{40}}}$, $\underline{\underline{B = \frac{-1}{20}}}$

(v) gen Form: $\boxed{y(t) = c_1 e^{-2t} + c_2 e^{6t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t)}$

EX Solve $y'' - 4y' - 12y = 2t^3 - t + 3$

(i) $y_c = c_1 e^{-2t} + c_2 e^{6t}$

(ii) Form: $\boxed{Y_p(t) = At^3 + Bt^2 + Ct + D}$

(iii) ODE: $(6At+2B) - 4(3At^2+2Bt+C) - 12(At^3+Bt^2+Ct+D) = 2t^3 - t + 3$

(iv) gather like terms:

$$(-12A)t^3 + (-12A-12B)t^2 + (6A-8B-12C)t + (2B-4C-12D) = 2t^3 - t + 3$$

match powers:

t^3: $-12A = 2 \to A = -1/6$

t^2: $-12A - 12B = 0 \to B = 1/6$

t^1: $6A - 8B - 12C) = -1 \to C = -1/9$

t^0: $2B - 4C - 12D = 3 \to D = -5/27$

(v) gen. solution:

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{6t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}}$$

So far we can summarize our forms:

$g(t)$	$Y_p(t)$ form
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} deg polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$

* lets get more complicated

$\boxed{\text{Ex}}$ Solve $y'' - 4y' - 12y = te^{4t}$

(i) $y_c = c_1 e^{-2t} + c_2 e^{6t}$

(ii) Form: $\boxed{Y_p = (At + B)e^{4t}}$ Y_p'' Y_p'

(iii) ODE: $\left(e^{4t}(16At + 16B + 8A)\right) - 4\left(e^{4t}(4At + 4B + A)\right)$
$$-12\left(e^{4t}(At + B)\right) = te^{4t}$$
Y_p

(iv) gather:
$$(16A - 16A - 12A)te^{4t} + (16B + 8A - 16B - 4A - 12B)e^{4t} = te^{4t}$$
$$\Rightarrow \quad -12At\,e^{4t} + (4A - 12B)e^{4t} = te^{4t}$$

match terms:

te^{4t} : $-12A = 1$ $\rightarrow A = -1/12$

e^{4t} : $4A - 12B = 0$ $\quad B = -1/36$

(v) gen. Soln:
$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{6t} - \frac{1}{12} te^{4t} - \frac{1}{36} e^{4t}}$$

⊛ The table expansion to

$g(t)$	$Y_p(t)$
$(n^{th}\text{ polynomial}) \cdot e^{\beta t}$	$(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) \cdot e^{\beta t}$
$(n^{th}\text{ poly}) \cdot (a\sin(\beta t) + b\cos(\beta t))$	$(A_n t^n + \cdots A_1 t + A_0) \cdot \sin\beta t + (B_n t^n + \cdots + B_1 t + B_0) \cdot \cos(\beta t)$

To be Cont.

II Superposition Principle

THM: let Y_{p_1} be a solution to

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and let Y_{p_2} be a solution to

$$y'' + p(t)y' + q(t)y = g_2(t)$$

then $Y_{p_1} + Y_{p_2}$ is the solution to the ODE

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$$

EX from the previous three examples, the particular solution to

$$y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t}$$

is

$$Y_p = -\frac{3}{7}e^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)e^{4t}$$

Don't forget

$$y_{gen} = c_1 e^{-2t} + c_2 e^{6t} + (\quad)$$

Now we apply I. Conditions.

Ex State the form of the particular soln Y_p if the driving function, $g(t)$ is

(a) $g(t) = 4\cos(6t) + 9\sin(3t)$

$$\boxed{Y_p(t) = A\cos(6t) + B\sin(6t) + C\cos(3t) + D\sin(3t)}$$

(b) $g(t) = 10e^{t} - 5te^{-8t} + 2e^{-8t}$

$$\boxed{Y_p(t) = Ae^{t} + (Bt + C)e^{-8t}}$$

(c) $g(t) = t^2\cos t - 5\sin t$

$$\boxed{Y_p(t) = (At^2 + Bt + C)\cos t + (Dt^2 + Et + F)\sin t}$$

(d) $g(t) = 5e^{-3t} + e^{-3t}\cos 6t - \sin 6t$

$$\boxed{Y_p(t) = Ae^{-3t} + (B\cos 6t + C\sin 6t)e^{-3t} + D\cos 6t + E\sin 6t}$$

III when $g(t)$ contains the complimentary solution, or part thereof

$\boxed{\text{Counter ex}}$ Find a particular soln, Y_p, for the ODE:

$$y'' - 4y' - 12y = e^{6t}$$

(i) $y_c = c_1 e^{-2t} + c_2 e^{6t}$

(ii) $Y_p = A e^{6t}$

(iii) $Y_p' = 6Ae^{6t}$, $Y_p'' = 36Ae^{6t}$

ODE: $(36Ae^{6t}) - 4(6Ae^{6t}) - 12(Ae^{6t}) = e^{6t}$

(iv) $\underbrace{36A - 24A - 12A}_{} = 1$

$0A = 1$ $\times$

Q: where did we go wrong? Ans: $g(t)$ has a term in it from y_c.

We need to modify Ae^{6t} to allow us to match.

We can fix the problem by multiplying Ae^{6t} by "t"

(ii) $\boxed{Y_p = Ate^{6t}}$ mult. by "t" forms a new Lin. Indep. function from e^{6t}

(iii) $Y_p' = Ae^{6t} + 6Ate^{6t}$, $Y_p'' = 6Ae^{6t} + 6A \cdot 1 \cdot e^{6t} + 6At(6e^{6t})$

$Y_p'' = 12Ae^{6t} + 36Ate^{6t}$

ODE: $(12Ae^{6t} + 36te^{6t}) - 4(Ae^{6t} + 6Ate^{6t}) - 12(Ate^{6t}) = e^{6t}$

(iv) match: e^{6t} : $12A - 4A = 1$ $\longrightarrow$ $A = 1/8$

te^{6t} : $\underbrace{36A - 24A - 12A}_{0} = 0$

(v) gen $\boxed{y_g = c_1 e^{-2t} + c_2 e^{6t} + \frac{1}{8} t e^{6t}}$ $\longleftarrow$ $\boxed{Y_p(t) = \frac{1}{8} t e^{6t}}$

Modify step (ii) to include these add'l steps

(a) create the standard Y_p form

(b) Compare Y_p with y_c

(c) if any terms differ by only a CONSTANT then add a "t" to that term.

$\boxed{ex}$ Form Y_p for $y'' + 3y' - 28y = 7t + e^{-7t} - 1$

(i) y_c: $r^2 + 3r - 28 = 0 \Rightarrow (r+7)(r-4) = 0 \quad r = -7, 4$

$$y_c = c_1 e^{-7t} + c_2 e^{4t}$$

(ii) Form Y_p: (a) normally $Y_p = (At + B) + Ce^{-7t}$

(b) compare: yes e^{-7t} shows up both times AND they diff by only a constant.

(C) add t to the e^{-7t} term in Y_p

$$\boxed{Y_p = (At + B) + Cte^{-7t}}$$

Ex Form Y_p for $y'' - 100y = 9t^2 e^{10t} + \cos(t) - t\sin(t)$

(i) y_c: $r^2 - 100 = 0$ $\quad r = \pm 10$ $\quad y_c = C_1 e^{-10t} + C_2 e^{10t}$

(ii) Y_p: (a) $Y_p = (At^2 + Bt + C)e^{10t} + (Dt + E)\cos(t) + (Ft + G)\sin(t)$

(b) compare y_c & Y_p: yes $\underline{Ce^{10t}}$ is only a constant from $\underline{C_2 e^{10t}}$

(c) remedy: mult by t:

$$\boxed{Y_p = t(At^2 + Bt + C)e^{10t} + (Dt + E)\cos(t) + (Ft + G)\sin(t)}$$

(iii) etc.. etc.

Ex Form Y_p for $4y'' + y = e^{-2t}\sin(t/2) + 6t\cos(t/2)$

(i) y_c: $\quad 4r^2 + 1 = 0$ $\quad r = \pm\sqrt{-\frac{1}{4}} = \pm\frac{i}{2}$, $y_c = C_1\cos(\frac{t}{2}) + C_2\sin(\frac{t}{2})$

(ii) Y_p: (a) normal guess $Y_p = Ae^{-2t}\sin(\frac{t}{2}) + Be^{-2t}\cos(\frac{t}{2}) + (Ct + D)\cos(\frac{t}{2})$
$(Et + F)\sin(t/2)$

(b) Compare Y_p to y_c: yes $D\cos(\frac{t}{2})$ & $F\sin(\frac{t}{2})$ differ by only a constant so insert "t"

(c) insert "t"

$$\boxed{Y_p = Ae^{-2}\sin(\tfrac{t}{2}) + Be^{-2}\cos(\tfrac{t}{2}) + t(Ct + D)\cos(\tfrac{t}{2}) + t(Et + F)\sin(\tfrac{t}{2})}$$

Ex Find Y_p for $y'' + 8y' + 16y = e^{-4t} + (t^2+5)e^{-4t}$

(i) y_c: $r^2 + 8r + 16 = 0 \rightarrow (r+4)^2 = 0$ $r = -4, -4$ double

$y_c = c_1 e^{-4t} + c_2 t e^{-4t}$

can consolidate

(ii) Y_p: (a) normal $\boxed{Y_p = Ae^{-4t} + (Bt^2 + Ct + D)e^{-4t}}$

$\boxed{Y_p = (Bt^2 + Ct + D)e^{-4t}}$

(b) Compare y_c with Y_p: yes De^{-4t} matches $c_1 e^{-4t}$

(c) $\boxed{Y_p = t(Bt^2 + Ct + D)e^{-4t}}$

(d) Q: Does the new Y_p still have terms that look like y_c

A: yes tDe^{-4t} match $c_2 t e^{-4t}$

Solution: Add another "t"

$\boxed{Y_p = t^2(Bt^2 + Ct + D)e^{-4t}}$

(e) Ask. now do any terms in Y_p differ by only a constant from any terms in y_c? None now.

(ii) (a) Form Y_p normally

(b) compare Y_p with y_c: Do any terms differ by only a constant?

(c) if so, mult. that term by t

(d) repeat (b) and (c) as needed.

(4.4 and 4.5 are now completed, do the HW.)