4.4 Undetermined Coefficients

LAST section we focused on various Solutions to HLODE with constant coefficients.

Today we focus on such equations with a non-homogenous part present, $g(x)$

- Recall the gen. soln is

$$y(x) = \underbrace{y_{\text{complimentary}}}_{\text{LHODE}=0} + \underbrace{y_{\text{particular}}}_{\text{LODE}=g}$$

Note, these functions have <u>derivatives</u> that end up looking like themselves:

- polynomials: $\sum_i a_i x^i$ {includes constants}
- exponentials: $\sum a_j e^{jx}$
- trig: $\sum_i a_j \cos(A_j x) + \sum b_j \cos(B_j x)$

- cross functions

$$x^m e^{px}, \; e^{px}\cos(bx), \ldots$$

Ex $y''+4y'-2y=\underbrace{2x^2-3x+6}_{g(x)}$

$m^2+4m-2=0 \Rightarrow m=-2\pm\sqrt{6}$

• solve the HLODE: $\boxed{y_c(x)=c_1e^{(-2-\sqrt{6})x}+c_2e^{(-2+\sqrt{6})x}}$

• Lets now assume that the full ODE has a solution that looks like itself.

$y_p(x)=Ax^2+Bx+C$

• plug it in: $y_p'=2Ax+B$, $y_p''=2A$

$y''+4y'-2y=2x^2-3x+6$

$(2A)+4(2Ax+B)-2(Ax^2+Bx+C)=2x^2-3x+6$

$x^2: \; -2A=2 \Rightarrow A=-1$

$x^1: \; 8A-2B=-3 \Rightarrow 8(-1)-2(B)=-3 \Rightarrow B=-5/2$

$x^0: \; 2A+4B-2C=6 \Rightarrow 2(-1)+\overset{2}{4}\left(-\frac{5}{\cancel{2}}\right)-2C=6 \Rightarrow -2-10-2C=6$

$C=-9$

So $\boxed{y_p=-x^2-\frac{5}{2}x-9}$

∴ the gen soln $\boxed{y_g(x)=c_1e^{(-2-\sqrt{6})x}+c_2e^{(-2+\sqrt{6})x}-x^2-5/2\,x-9}$

$\boxed{\text{Ex}}$ $y'' - y' + y = 2\sin 3x + 0\cos(3x)$

$$\hookrightarrow y_p = A\cos(3x) + B\sin(3x)$$
$$y_p' = -3A\sin 3x + 3B\cos 3x$$
$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

- plug into

$$(-9A\cos(3x) - 9B\sin(3x)) - (-3A\sin(3x) + 3B\cos(3x)) + (A\cos(3x) + B\sin(3x)) = 2\sin(3x) + 0\cos(3x)$$

$\cos(3x)$: $-8A - 3B = 0 \to B = -\frac{8}{3}A$ $\to \boxed{B = -\frac{16}{73}}$

$\sin(3x)$: $3A - 8B = 2 \to (3A - 8(-\frac{8}{3}A) = 2)\,3$

$$9A + 64A = 6$$
$$73A = 6 \Rightarrow \boxed{A = \frac{6}{73}}$$

$$\boxed{y_p = \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x)}$$

$\boxed{\text{EX}}$ $y'' - 2y' - 3y = \underbrace{4x-5}_{g_1(x)} + \underbrace{6xe^{2x}}_{g_2(x)}$

$m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

- complimentary soln: zeros $m = -1, 3$ $\boxed{y_c(x) = c_1 e^{-x} + c_2 e^{3x}}$

- particular: $g_1(x) = 4x - 5 \to y_{p_1} = Ax + B$

$g_2(x) = 6xe^{2x} \to y_{p_2}(x) = Cxe^{2x} + De^{2x}$

- Derivates $y_{p_1} = Ax + B, \; y_{p_1}' = A, \; y_{p_1}'' = 0$

$y_{p_2} = Cxe^{2x} + De^{2x}, \; y_{p_2}' = Ce^{2x} + C2xe^{2x} + 2De^{2x}$

$y_{p_2}'' = 2Ce^{2x} + 4De^{2x} + 2Ce^{2x} + 4Cxe^{2x}$

- Insert:

$\boxed{y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}}$ ODE (again)

$\boxed{y_{p_1}}$

$(0) - 2(A) - 3(Ax+B) = 4x - 5$ $\begin{cases} x^1: -3A = 4 \\ x^0: -2A - 3B = -5 \end{cases}$ $\boxed{A = -4/3}$

$\Rightarrow \boxed{B = +\frac{23}{9}}$

$\boxed{y_{p_1}(x) = -\frac{4}{3}x + \frac{23}{9}}$

$\boxed{y_{P_2}}$ $y'' - 2y' - 3y = 6xe^{2x} + 0 \cdot e^{2x}$

$$\boxed{y_{P_2} = Cxe^{2x} + De^{2x}}$$
$$y_{P_2}' = Ce^{2x} + C2xe^{2x} + 2De^{2x}$$
$$y_{P_2}'' = 2Ce^{2x} + 4De^{2x} + 2Ce^{2x} + 4Cxe^{2x}$$

- Insert into the NHLODE:

$$(2Ce^{2x} + 4De^{2x} + 2Ce^{2x} + 4Cxe^{2x}) - 2(Ce^{2x} + 2Cxe^{2x} + 2De^{2x})$$
$$-3(Cxe^{2x} + De^{2x}) = 6xe^{2x}$$

xe^{2x} : $+\cancel{4C} - \cancel{4C} - 3C = 6 \Rightarrow \boxed{C = -2}$

e^{2x} : $\cancel{2C} - \cancel{2C} + \cancel{4D} - \cancel{4D} + 2C + (-3D) = 0 \Rightarrow -4 - 3D = 0$

$-3D = 0 + 4$

$D = -4/3$

$$\boxed{y_{P_2}(x) = -2xe^{2x} - \frac{4}{3}e^{2x}}$$

* Join all pieces

4?

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{3x} + \left(-\frac{4}{3}x + \frac{23}{9}\right) + \left(-2xe^{2x} - \frac{4}{3}e^{2x}\right)}$$

* what if our $g(x)$ produces a guess of y_p that Contain a part of the complimentary soln.

$\boxed{\text{Ex}}$ $\underbrace{y''-5y'+4y=8e^x}$ $\rightarrow$ $\boxed{y_p = Ae^x,\ y_p' = Ae^x,\ y_p'' = Ae^x}$

$m^2-5m+4=0$

$(m-1)(m-4)=0$ $\rightarrow$ $y_c = c_1e^x + C_2e^{4x}$

$m=1,\ m=4$

math speak Contradiction

Insert: $(A\cancel{e^x}) - 5(A\cancel{e^x}) + 4(A\cancel{e^x}) = 8\cancel{e^x}$

$0 \cdot A = 8$

We suggest we assume $\boxed{y_p = A\underline{\underline{x}} \cdot e^x \quad \{\text{add } x\}}$

$\Rightarrow$ $3Ae^x = 8e^x \Rightarrow \boxed{A = -\frac{8}{3}} \Rightarrow \boxed{y_p(x) = -\frac{8}{3}xe^x}$

[To Be continued (4.3 only, due on Mon) Skip 4.5 & 4.6]

If $g(x)$ equals:	Then pick $y_p =$:
1. $\underbrace{a_0x^n + a_1x^{n-1} + \cdots + a_n}_{P_n(x)}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
2. $P_n(x)e^{\alpha x}$	$x^s\underbrace{(A_0x^n + A_1x^{n-1} + \cdots + A_n)}e^{\alpha x}$
3. $P_n e^{\alpha x}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^s(A_0x^n + \cdots A_n)e^{\alpha x}\cos(\beta x) + x^s(B_0x^n + \cdots B_n)e^{\alpha x}\sin(\beta x)$

Pick $s = 0, 1$ or 2 such that no term in $y_p \cdot x^s$ is a solution of the corresponding HLODE.

4.4 cont. e^{mx}

EX: Find the gen. soln: $\boxed{y'' - 3y' - 4y = \underbrace{2e^{-x}}_{g(x)}}$

1. Solve for y_c: $m^2 - 3m - 4 = 0$

$(x-4)(x+1) = 0 \rightarrow m = 4, -1$

$$\boxed{y_c = C_1 e^{-x} + C_2 e^{4x}}$$

2. Is $g(x)$ in the family of driving functs we are studying? (polys, exp. sin & cos.)

3. Do we need to break $g(x)$ into parts? No

4. Select the form from table: {Keep in mind that $g(x)$ might have pieces that look like y_c}

* we do have a piece of $g(x)$ that matches a piece of y_c. *

So we will pick $\boxed{y_p = A x^{s} e^{-x}}$ $s=1$ → $\boxed{y_p = Axe^{-x}}$

$y_p' = -Axe^{-x} + Ae^{-x}$

5. $y_p'' = -Ae^{-x} + Axe^{-x} - Ae^{-x}$

Insert into the ODE:

$$\Rightarrow (-Ae^{-x} + \cancel{Axe^{-x}} - Ae^{-x}) - 3(-\cancel{Axe^{-x}} + Ae^{-x}) - 4(\cancel{Axe^{-x}}) = 2e^{-x}$$

$A = -\frac{2}{5}$

6. $\boxed{y_{gen} = C_1 e^{-x} + C_2 e^{4x} - \frac{2}{5} x e^{-x}}$

EX: $$\boxed{y'' - 6y' + 9y = \overbrace{\underbrace{6x^2+2}_{g_1} \underbrace{-12e^{3x}}_{g_2}}^{g}}$$

1. $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow \boxed{y_c = c_1 e^{3x} + c_2 x e^{3x}}$

2. Note that g_1 & g_2 have the forms we have studied.

3. We can split $g(x) = g_1(x) + g_2(x) = \underbrace{(6x^2+2)}_{g_1} + \underbrace{(-12e^{3x})}_{g_2}$

4. Table: $\boxed{y_{p_1} = Ax^2 + Bx + C}$ & $\boxed{y_{p_2} = Dx^2 e^{3x}}$ $\quad s=2$

5. $y_{p_1}' = 2Ax + B \qquad y_{p_2}' = D[2xe^{3x} + x^2 3e^{3x}]$

$y_{p_1}'' = 2A \qquad y_{p_2}'' = D[2e^{3x} + 2x \cdot 3e^{3x} + 2x3e^{3x} + x^2 3^2 e^{3x}]$

Insert into the ODE's:

① ODE: $(2A) - 6(2Ax+B) + 9(Ax^2+Bx+C) = 6x^2+2$

$x^2 : 9A = 6 \Rightarrow \boxed{A = 2/3}$

$x^1 : -12A + 9B = 0 \rightarrow \boxed{B = 8/9}$

$x^0 : 2A - 6B + 9C = 2 \rightarrow \boxed{C = 2/3}$

$\boxed{y_{p_1} = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3}}$

② ODE: $D[2e^{3x} + 6xe^{3x} + 2x3e^{3x} + 3^2 x^2 e^{3x}] - 6D[2xe^{3x} + x^2 3e^{3x}] + 9[Dx^2 e^{3x}] = -12e^{3x}$

$\div e^{3x} \Rightarrow D[2 + 12x + 9x^2] - 6D[2x + 3x^2] + 9Dx^2 = -12$

$\Rightarrow x^2 : (9 - 18 + 9) \cdot D = 0 \Rightarrow 0 \cdot D = 0$ inconclusive

$x^1 : (12 - 12) \cdot D = 0 \Rightarrow 0 \cdot D = 0$

$x^0 : 2D = -12 \Rightarrow \boxed{D = -6}$

$\boxed{y_{p_2} = -6x^2 e^{3x}}$

⑥ $$\boxed{y_{gen} = C_1 e^{3x} + C_2 x e^{3x} + \left(\frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3}\right) - 6x^2 e^{3x}}$$

⊛ Higher order eqns

$\boxed{EX}$ Solve $\boxed{y''' + y'' = e^x \cos x}$

$c_1 e^{0x} + c_2 x e^{0x}$

1. HLODE: $m^3 + m^2 = 0 \to (m+1)(m^2) = 0$, $\boxed{m = 0, 0, -1}$

$$\boxed{y_c = c_1 + c_2 x + c_3 e^{-x}}$$

$c_3 e^{-x}$

2. $g = e^x \cos x$ is allowed.
3. No need to break $g(x)$ into pieces
4. Form: $\boxed{y_p = A e^x \cos x + B e^x \sin x} = e^x [A c x + B s x]$
5. Insert: $y_p' = e^x [A \cos x + B \sin x] + e^x [-A \sin x + B \cos x]$

$$\left\{\begin{array}{l} y_p'' = e^x \{(A+B)\cos x + (B-A)\sin x\} + e^x \{-(A+B)\sin x + (B-A)\cos x\} \\ y_p'' = e^x \{2B \cos x - 2A \sin x\} \end{array}\right.$$

$$\left\{\begin{array}{l} y_p''' = e^x [2B \cos x - 2A \sin x] + e^x [-2B \sin x - 2A \cos x] \\ y_p''' = e^x [(2B - 2A)\cos x - (2A + 2B)\sin x] \end{array}\right.$$

ODE:

$$\left(e^x [(2B-2A)\cos x - (2A+2B)\sin x]\right) + \left(e^x [2B\cos x - 2A \sin x]\right) = e^x \cos x$$

$$\left.\begin{array}{l} \cos x: (2B - 2A) + 2B = 1 \to 4B - 2A = 1 \\ \sin x: -(2A + 2B) - 2A = 0 \to -2B - 4A = 0 \end{array}\right\} \Rightarrow \left(\begin{array}{cc|c} -2 & 4 & 1 \\ -4 & -2 & 0 \end{array}\right) \div -2$$

$$\Rightarrow \left(\begin{array}{cc|c} -2 & 4 & 1 \\ 2 & 1 & 0 \end{array}\right) +$$

$$\Rightarrow \left(\begin{array}{cc|c} 0 & 5 & 1 \\ 2 & 1 & 0 \end{array}\right)$$

$$\boxed{B = 1/5, A = -1/5}$$

6. $\boxed{y_{gen} = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{5} e^{+x} \cos x + \frac{1}{5} e^x \sin x}$