Higher Order ODEs

First, a Summary of the Three Cases

With these three options lets do basic examples:

$\boxed{\text{EX}}$ (from text)

case:

I (a) $2y''-5y'-3y=0 \Rightarrow 2m^2-5m-3=0 \Rightarrow (2m+1)(m-3)=0$

$m=-1/2,\ m=3$

$\Rightarrow \boxed{y(x)=c_1e^{-x/2}+c_2e^{3x}}$

II (b) $y''-10y'+25y=0 \Rightarrow m^2-10m+25=0 \Rightarrow (m-5)(m-5)=0$

$m=5, 5$

$\Rightarrow \boxed{y(x)=c_1e^{5x}+c_2\underline{x}e^{5x}}$

III (c) $y''+y'+y=0 \Rightarrow m^2+m+1=0 \Rightarrow m=\frac{-1\pm\sqrt{(-1)^2-4(1)(1)}}{2\cdot 1}=\underbrace{-\frac{1}{2}}_{\alpha}\pm\underbrace{\frac{\sqrt{3}}{2}}_{\beta}i$

$\Rightarrow \boxed{y(x)=c_1e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)}$

$\boxed{\text{EX}}$ $y''-k^2y=0$

- Hanging-Cables
- Char. eqn: $m^2-k^2=0$

  $m=\pm k$
- soln: $\boxed{y(x)=c_1e^{kx}+c_2e^{-kx}}$

or, since $\cosh(kx)=\frac{e^{kx}+e^{-kx}}{2}$

and $\sinh(kx)=\frac{e^{kx}-e^{-kx}}{2}$

we can recast the soln...

soln: $\boxed{y(x)=c_1\cosh(kx)+c_2\sinh(kx)}$

Hyperbolic

$y''+k^2y=0$

- spring-mass problem
- characteristic eqn: $m^2+k^2=0$

  $m=\pm ki$

  $\alpha=0,\ \beta=k$
- soln: $\boxed{y=c_1\cos kx+c_2\sin kx}$

Trig functions

(*) Higher Order Eqns:

The ODE $\sum_{i=0}^{n} a_i y^{(i)} = 0$ has a characteristic eqn

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_2 m^2 + a_1 m + a_0 = 0$$

For Distinct roots:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

For m_1 having a multiplicity "k":

$$y_c(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \dots + c_k x^{n-1} e^{m_1 x}$$

For a combination of multiple real roots and some distinct roots, we use a corresponding combination of these two.
{use synthetic division to obtain and analyze roots}

KEY

EX $3y''' + 5y'' + 10y' - 4y = 0 \Rightarrow 3m^3 + 5m^2 + 10m - 4 = 0$

$\exists$ 3 roots...

$\hookrightarrow \pm 1, \pm 3$ $\quad$ $\hookrightarrow \pm 1, \pm 2, \pm 4$

For Rational zeros, if they exist are from this

Pool of rational zeros: $\{-4, -2, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2, 4\}$
Lowest $\quad$ Highest

are ratios of the caboose factors over the leading

$\frac{1}{3}$	3	5	10	-4
		1	2	4
	3	6	12	0

We find one $\quad$ $\frac{1}{3}$ is a root

The reduced polynomial: $3m^2 + 6m + 12 = 0 \Rightarrow m^2 + 2m + 4 = 0$

$$\Rightarrow \begin{cases} m_2 = -1 + \sqrt{3}\, i \\ m_3 = -1 - \sqrt{3}\, i \end{cases}$$

gen soln: $m_1 = \frac{1}{3}$, $m_2 = -1 + \sqrt{3} i$, $m_3 = -1 - \sqrt{3} i$

$$\Rightarrow \quad y(x) = c_1 e^{\frac{1}{3} x} + c_2 e^{-x} \cos \sqrt{3}\, x + c_3 e^{-x} \sin \sqrt{3}\, x$$

$\boxed{EX}$: $y'''' + 2y'' + y = 0 \Rightarrow m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0$ 5

$\Rightarrow$ $m_1 = i, -i$ each w/ multiplicity 2,

"i" : $\cos x$, $x\cos x$ "$-i$" : $\sin x$, $x\sin x$

$\Rightarrow$ gen soln $\boxed{y(x) = C_1\cos x + C_2\sin x + C_3 x\cos x + C_4 x\sin x}$

Finally we could have some complex conjugate pairs, and maybe even multiple identical pairs.

We will then add to the other two configurations this type of soln: For multiple pairs of identical complex conjugates:

$\begin{cases} m_{1,3} = \alpha + i\beta \quad (\beta > 0) \\ m_{2,4} = \alpha - i\beta \quad (\beta > 0) \end{cases}$ conjugate

$\Rightarrow$

$$\underbrace{C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x}_{m_1 \,\&\, m_2} + \underbrace{C_3 x e^{\alpha x}\cos\beta x + C_4 x e^{\alpha x}\sin\beta x}_{m_3 \text{ and } m_4}$$

$\boxed{EX}$: $2\frac{d^5y}{dx^5} - 7\frac{d^4y}{dx^4} + 12\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} = 0$

$2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$

$m^2(\underset{\text{engine}}{2}m^3 - 7m^2 + 12m + \underset{\text{caboose}}{8}) = 0 \Rightarrow \boxed{m_1, m_2 = 0} \longrightarrow C_1 + C_2 x$

- If there is a rational zero(s) it is a multiple of the caboose's factors over the engine's factors: $\left\{\frac{1,2,4,8}{1,2} = 1, 2, 4, 8; \frac{1}{2}, 1, 2, 4; = \pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8\right\}$

Start with

```
1 | 2  -7  12   8
  |     2  -5   7
  ---------------
    2  -5   7  15 <

-1 | 2  -7  12   8
   |    -2   9  -21
   ---------------
lower bound →  2  -9  21  -13

1/2 | 2  -7  12   8
    |     1  -3  4.5
    ---------------
      2  -6   9  12.5

-1/2 | 2  -7  12   8
     |    -1   4  -8
     ---------------
       2  -8  16   0
```

← A zero!! $m = -1/2$

now $m^2(m + \frac{1}{2})(2m^2 - 8m + 16) = 0$

$m^2 - 4m + 8 = 0 \Rightarrow m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 8}}{2\cdot 1}$

$m = \frac{4 \pm \sqrt{16 - 32}}{2}$

$\boxed{m = 2 \pm 2i}$

$y(x) = C_1 + C_2 x + C_3 e^{-x/2} + C_4 e^{2x}\cos 2x + C_5 e^{2x}\sin 2x$

EX: #48

$$y''' + 2y'' - 5y' - 6y = 0 \qquad \begin{cases} y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 1 \end{cases}$$

$$m^3 + 2m^2 - 5m - 6 = 0 \longrightarrow \text{rational root candidates} \{\pm 1, \pm 2, \pm 3, \pm 6\}$$

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -5 & -6 \\ & & 2 & 4 & -1 \\ \hline & 1 & 4 & -1 & -7 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array} \checkmark$$

$$\Rightarrow m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$\Rightarrow m = -1, -3, 2$$

gen soln: @ $x = 0$

$$\Rightarrow \boxed{y(x) = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x}} \rightarrow \boxed{0 = c_1 + c_2 + c_3}$$

$$y' = -c_1 e^{-x} - 3c_2 e^{-3x} + c_3 2e^{2x}$$

$$y'(0) = -c_1 - 3c_2 + 2c_3 \Rightarrow \boxed{0 = -c_1 - 3c_2 + 2c_3}$$

$$y'' = c_1 e^{-x} + 9c_2 e^{-3x} + 4c_3 e^{2x} \quad \boxed{1 = c_1 + 9c_2 + 4c_3}$$

Solving the 3x3 system of coefficients

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ -1 & -3 & 2 & 0 \\ 1 & 9 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 0 & -2 & 3 & 0 \\ -1 & -3 & 2 & 0 \\ 0 & 6 & 6 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 0 & -2 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 1/6 \end{array}\right) * -3; *2$$

$$\left(\begin{array}{rrr|r} 0 & 0 & 5 & 1/3 \\ 1 & 0 & -5 & -1/2 \\ 0 & 1 & 1 & 1/6 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 0 & 0 & 1 & 1/15 \\ 1 & 0 & 0 & -1/6 \\ 0 & 1 & 1 & 1/6 \end{array}\right) * -1 \rightarrow \left(\begin{array}{rrr|r} 0 & 0 & 1 & 1/15 \\ 1 & 0 & 0 & -1/6 \\ 0 & 1 & 0 & \frac{1}{6} - \frac{1}{15} \end{array}\right)$$

$\div 5$

$$\Rightarrow c_3 = 1/15, \quad c_1 = -\frac{1}{6}, \quad c_2 = \frac{1}{6} - \frac{1}{15} = \frac{5}{30} - \frac{2}{30} = \frac{3}{30} = \frac{1}{10}$$

$$\boxed{y(x) = -\frac{1}{6} e^{-x} + \frac{1}{10} e^{-3x} + \frac{1}{15} e^{2x}} \checkmark$$