$\boxed{4.3c}$ 2nd order Lin. Const. Coeff, Homog, ODE with complex roots in its characteristic eqn.

Recall $\boxed{ay'' + by' + cy = 0}$

we assumed this type of soln: $y = ce^{rt}$

which yeilded the ODE's characteristic eqn:

$$\boxed{ar^2 + br + c = 0}$$

• 3 cases: order of coverage

- 4.3a I. $r_1 \neq r_2 \in \mathbb{R}$ (i)
- 4.3b II. $r_1 = r_2 \in \mathbb{R}$ (iii)
- $\boxed{4.3c}$ III. $r_1, r_2 \in \mathbb{Z}$ complex numbers (conjugates) (ii)

Case III: the roots of $ar^2 + br + c = 0$ are imaginary or complex numbers. $r = \boxed{\lambda + \mu i}$ — complex number; λ real, μ imag.

We know from algebra if "r" is complex and is a zero of a polynomial, so is its conjugate a root of the polynomial. So the two solutions are

$$\boxed{y_1 = c_1 e^{(\lambda + \mu i)t}, \quad y_2 = c_2 e^{(\lambda - \mu i)t}}$$

We focus only on y_1 (since y_2 is the complex conj. the solution of y_1 immediately tells us y_2)

• split the exponetial up:

$$y_1 = c_1 e^{\lambda t} e^{\mu i t}$$

Now Recall Euler's Famous Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

{BTW: if $\theta = \pi$ $\Rightarrow e^{i\pi} = -1 + i0$ }

So $y_1(t) = ce^{\lambda t}e^{\mu i t} = \boxed{ce^{\lambda t}[\cos(\mu t) + i\sin(\mu t)]}$

then $\boxed{y_2 = ce^{\lambda t}[\cos(\mu t) - i\sin(\mu t)]}$

{Both real and imaginary satisfy the ODE: $ay'' + by' + cy = 0$ }

• Let $y_1 = y_r + iy_i$ and let $y_2 = y_r - iy_i$

$y_1' = y_r' + iy_i'$ $\qquad y_2' = y_r' - iy_i'$

$y_1'' = y_r'' + iy_i''$ $\qquad y_2'' = y_r'' - iy_i''$

Insert into the ODE: $ay'' + by' + cy = 0$ — and separate into real and imaginary

$$\Rightarrow (ay_r'' + by_r' + cy_r) \pm i(ay_i'' + by_i' + cy_i) = 0 + 0i$$

Here we observed the real and imag. parts are both solutions

• Now note $y_1(t) = e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)]$

$+\ y_2(t) = e^{\lambda t}[\cos(\mu t) - i\sin(\mu t)]$

so $y_1 + y_2 = e^{\lambda t}2\cos(\mu t) + i0$

since each y_1 & y_2 are solutions.

so is $y_1 + y_2$. but $\boxed{\frac{y_1 + y_2}{2} = y_{real}}$

$\Rightarrow \boxed{y(t) = e^{\lambda t}\cos(\mu t)}$ is a solution to the ODE

We repeat but now subtract: ③

$$y_1(t) = e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)]$$
$$- \quad y_2(t) = e^{\lambda t}[\cos(\mu t) - i\sin(\mu t)]$$
$$y_1 - y_2 = 0 + i2e^{\lambda t}\sin(\mu t)$$

but $\boxed{y_{im} = \frac{y_1 - y_2}{2}}$ is also a solution to the ODE.

- every derivative of $ie^{\lambda t}\sin(\mu t)$ has an "i"

we can factor out the "i" and $\div i$ leaving

a new function (real):

$$\boxed{y(t) = e^{\lambda t}\sin(\mu t)}$$ Indep. solution

So we finally form the gen solution:

SUMMARY

For $\boxed{ay'' + by' + cy = 0}$ if the roots of $ar^2 + br + c = 0$ are complex then the solution of the ODE is

$$\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)}$$

- This is the general solution. No imaginary part is present or needed. In 4.2 we will introduce the Wronskian as a tool to test that these two functions are Lin. Indep and form a Fund. Soln.

EX $y'' + 16y = 0, \quad y(\frac{\pi}{2}) = -10, \quad y'(\frac{\pi}{2}) = 3$

(i) $r^2 + 16 = 0$

(ii) $r = +4i, -4i \rightarrow \lambda = 0, \mu = 4$

(iii) $y(t) = c_1 e^{0\cdot t}\cos(4t) + c_2 e^{0\cdot t}\sin(4t)$

or $\boxed{y(t) = c_1\cos(4t) + c_2\sin(4t)}$ gen. soln.

(iv) IC: $y(\frac{\pi}{2}) = c_1\cos(4\cdot\frac{\pi}{2}) + c_2\sin(4\cdot\frac{\pi}{2})$

$-10 = c_1\cdot 1 + c_2\cdot 0 \Rightarrow \boxed{c_1 = -10}$

$y'(t) = -4c_1\sin(4t) + 4c_2\cos(4t)$

$y'(\frac{\pi}{2}) = -4\cdot c_1\cdot 0 + 4\cdot c_2\cdot 1$

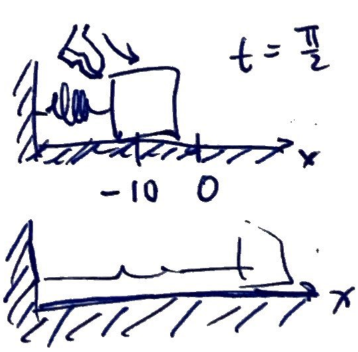
$3 = 4c_2 \rightarrow \boxed{c_2 = 3/4}$

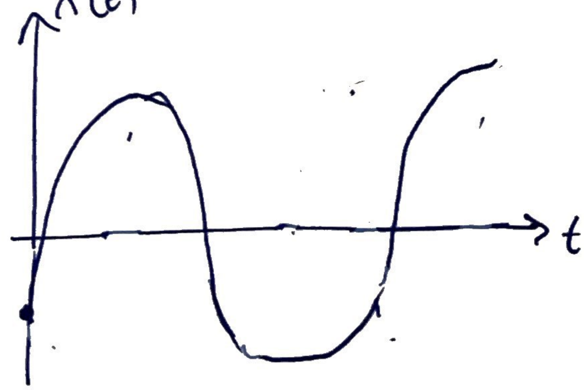
$\boxed{y(t) = -10\cos(4t) + \frac{3}{4}\sin(4t)}$ specific soln.

In physics:

no friction so no decay of amplitude

$\boxed{IC}$ we both displace & kick the mass





EX Solve the IVP

$$y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -8$$

↑ no displacement ↑ kick only

(i) gen soln.

Characteristic: $r^2 - 4r + 9 = 0$

Factor: $(r - 2 - \sqrt{5}i)(r - 2 + \sqrt{5}i) = 0$

$\Rightarrow$ gen. soln: $\boxed{y(t) = c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t)}$

(ii) IVP. $y(0) = c_1 e^{2\cdot 0}\cos\sqrt{5}\cdot 0 + c_2 e^{2\cdot 0}\sin(\sqrt{5}\cdot 0)$

$$0 = c_1 + c_2 \cdot 0 \Rightarrow \boxed{c_1 = 0}$$

Solution to date

$$y(t) = c_2 e^{2t}\sin(\sqrt{5}t)$$

$$y'(t) = c_2 2e^{2t}\sin(\sqrt{5}t) + c_2 e^{2t}\sqrt{5}\cos(\sqrt{5}t)$$

now @ $t = 0$

$$y'(0) = c_2 2e^{2\cdot 0}\cancelto{0}{\sin(\sqrt{5}\cdot 0)} + c_2 e^{2\cdot 0}\sqrt{5}\cos(\sqrt{5}\cdot 0)$$

$$-8 = c_2 \cdot 1 \cdot \sqrt{5} \cdot 1 \Rightarrow \boxed{c_2 = -8/\sqrt{5}}$$

(iii) Final soln

$$\boxed{y(t) = -\frac{8}{\sqrt{5}} e^{2t}\sin(\sqrt{5}t)}$$

$\boxed{\text{EX}}$ Solve $4y'' + 24y' + 37y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$

(i) $4r^2 + 24r + 37 = 0$

(ii) $r = -3 \pm \frac{1}{2} i$

(iii) $\boxed{y(t) = c_1 e^{-3t} \cos(t/2) + c_2 e^{-3t} \sin(t/2)}$

(iv) $y(\pi) = c_1 e^{-3\pi} \cos\left(\frac{\pi}{2}\right) + c_2 e^{-3\pi} \sin\left(\frac{\pi}{2}\right)$

$1 = 0 + c_2 e^{-3\pi} \cdot 1 \Rightarrow \boxed{c_2 = e^{3\pi}}$

$y'(t) = -3c_1 e^{-3t} \cos\left(\frac{t}{2}\right) - c_1 e^{-3t} \frac{1}{2} \sin\left(\frac{t}{2}\right)$

$\quad + -3e^{3\pi} e^{-3t} \sin\left(\frac{t}{2}\right) + \left(e^{3\pi} e^{-3t} \cos(t/2)\right)\frac{1}{2}$

$y'(\pi) = -3c_1 e^{-3\pi} \cancel{\cos}\left(\frac{\pi}{2}\right)^{0} - c_1 e^{-3\pi} \frac{1}{2} \sin\left(\frac{\pi}{2}\right)$

$\quad - 3e^{3\pi} e^{-3\pi} \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} e^{3\pi} e^{-3\pi} \cancel{\cos}(\pi/2)^{0}$

$0 = -\frac{c_1}{2} e^{-3\pi} \cdot 1 - 3 \cdot 1$

$\frac{e^{-3\pi}}{2} \cdot c_1 = -3 \longrightarrow \boxed{c_1 = -6e^{3\pi}}$

(v). $\boxed{y(t) = e^{-3(t-\pi)} \cos\left(\frac{t}{2}\right) - 6e^{-3(t-\pi)} \sin\left(\frac{t}{2}\right)}$