4.2 Reduction of Order, 2^{nd} order ODE Existence & Uniqueness

Lets go back to variable coefficients

$$p(t)y'' + q(t)y' + r(t)y = 0$$

Q: Given a soln to this ODE, y_1, can we find the second solution? y_2.

* We will assume that $\boxed{y_2(t) = v(t)\, y_1(t)}$, a modification to the known solution y_1

This is a bit of a stretch. From 4.1 we know if two solns exist for a 2^{nd} order ODE and they are linearly independent then that solution forms a set $\{y_1, y_2\}$ that we call the Fundamental Solution Set. That is where we then form the general solution:

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$$

and no other solutions need be sought after.

$\boxed{\text{EX}}$ We know that $y_1(t) = t^{-1}$ is a solution to the Euler-Cauchy ODE:

$$2t^{2}y'' + t^{1}y' - 3\,{}_{1}y = 0 \qquad t^{0}$$

{ when the power of the polynomial = the degree of derivatives for all terms in an ODE it is called An Euler-Cauchy ODE }

- Let $\boxed{y_2 = v(t)\,t^{-1}}$ $\leftarrow y_1$, $t > 0$

then
$$y_2' = -t^{-2}v + t^{-1}v'$$
and
$$y_2'' = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$
product rule 2

- Insert these into the ODE { combine order of derivatives }

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(-t^{-2}v + t^{-1}v'\right) - 3vt^{-1} = 0$$

- group orders of v:

$$(v'')(2t^2 \cdot t^{-1}) + v'(-4t^0 + t^0) + v\underbrace{\left(4t^{-1} - t^{-1} - 3t^{-1}\right)}_{=0 \text{ always.}} = 0$$

$$\Rightarrow \boxed{2tv'' - 3v' = 0}$$

Note that we do not see v by itself.

- So let $\boxed{w(t) = v'(t)}$ then $w'(t) = v''(t)$ and the above ODE in v becomes

$$\boxed{2tw' - 3w = 0}$$

"Reduction of order" from 2 to 1 derivative

- lets solve the ODE in w: use separation

$$2t\,dw = 3w\,dt \quad \div 2t \Rightarrow \boxed{\frac{dw}{w} = \frac{3}{2}\frac{dt}{t}}$$

- Integrate: $\ln|w| = \frac{3}{2}\ln|t| + c$

- Raise as powers of e:

$$|w| = |t|^{3/2}\cdot C \qquad \text{but } t>0$$

$$\Rightarrow \boxed{w(t) = Ct^{3/2}}$$

- Back out of our substitutions:

(i) $v(t) = \int w(t)\,dt$ since $w \equiv v'$

$$v(t) = \int Ct^{3/2}\,dt \Rightarrow \boxed{v(t) = Ct^{5/2} + k}$$

- But (ii) $y_2 = v \cdot y_1$ so

$$y_2 = [Ct^{5/2} + k]\,t^{-1}$$

$$\Rightarrow \boxed{y_2(t) = Ct^{3/2} + kt^{-1}} \; ; \; y_1(t) = t^{-1}$$

same

- So the gen. solution

$$\boxed{y(t) = c_1 t^{3/2} + c_2 t^{-1}}$$ the gen. soln.

to $2t^2y'' + ty' - 3y = 0$

Def: let $W \equiv \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$ be the Wronskian of the two functions f & g.

$\boxed{\text{Thm: If } W \neq 0 \text{ then } f \text{ \& } g \text{ are linearly independent. That is, } f(t) \text{ is not a disguized } g(t).}$

$\boxed{\text{Ex}}$ Are e^t and $\sinh(t)$ Lin. Indp?

$$W = \begin{vmatrix} \sinh(t) & e^t \\ \cosh(t) & e^t \end{vmatrix} = e^t \sinh(t) - e^t \cosh(t)$$

$$= e^t\left(\frac{e^t - e^{-t}}{2}\right) - e^t\left(\frac{e^t + e^{-t}}{2}\right)$$

$= -1$, since not zero, yes $\sinh(t)$ & e^t are Lin. Indp.

$\boxed{\text{Ex}}$ Are $f = t^{-1}$ & $g = e^{-\ln(t)}$ Lin. Indp? $\quad e^{\ln(t^{-1})} \to t^{-1}$

$$f' = -\frac{1}{t^2} \qquad g' = e^{-\ln(t)} \cdot \left(-\frac{1}{t}\right)$$

$$W = \begin{vmatrix} t^{-1} & e^{-\ln(t)} \\ -\frac{1}{t^2} & \frac{-e^{-\ln(t)}}{t} \end{vmatrix} = \left(-\frac{e^{-\ln(t)}}{t^2}\right) - \left(-\frac{e^{-\ln(t)}}{t^2}\right)$$

$= 0$ Not Lin. InDep.

$\boxed{\text{EX}}$ are $\{\sinh(t), e^{-t}, e^{t}\}$ Lin. Indp? (5)

$$W = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \quad \text{for 3 functions}$$

So

$$W = \begin{vmatrix} \sinh(t) & e^{-t} & e^{t} \\ \cosh(t) & -e^{-t} & e^{t} \\ \sinh(t) & e^{-t} & e^{t} \end{vmatrix}$$

$$= +e^{t} \begin{vmatrix} \cosh(t) & -e^{-t} \\ \sinh(t) & e^{-t} \end{vmatrix} - e^{t} \underbrace{\begin{vmatrix} \sinh(t) & e^{-t} \\ \sinh(t) & e^{-t} \end{vmatrix}}_{0}$$

$$+ e^{t} \begin{vmatrix} \sinh(t) & e^{-t} \\ \cosh(t) & -e^{-t} \end{vmatrix}$$

$$= e^{t}\left[e^{-t}\cosh(t) + e^{-t}\sinh(t)\right] + e^{t}\left[-e^{-t}\sinh(t) - e^{-t}\cosh(t)\right]$$

$$= \cosh(t) + \sinh(t) - \sinh(t) - \cosh(t)$$

$= \boxed{0}$ the 3 functions $\{\sinh(t), e^{-t}, e^{t}\}$ are not Lin. Indp. That is because

$$\sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

i.e. $\sinh(t)$ is created from e^{-t} & e^{t}, or, as we say a Linear Combination of e^{-t} & e^{t}.

$\boxed{\text{Ex}}$ Back to our ODE $2t^2y'' + ty' - 3y = 0$

we were given $y_1 = t^{-1}$ and via reduction of order we found $y_2 = t^{3/2}$ $\{t^{-1}, t^{3/2}\}$

- Check Lin. Indep: for this solution set $\{t^{-1}, t^{3/2}\}$

$$W = \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2}t^{1/2} \end{vmatrix}$$

$$= (t^{-1})\left(\tfrac{3}{2}t^{1/2}\right) - (-t^{-2})(t^{3/2})$$

$$= \tfrac{3}{2}t^{-1/2} + t^{-1/2}$$

$\neq 0$ not identically zero, y_1 & y_2 are L.I.

- So we were justified in forming the gen. soln from these two functions.

$$y(t) = c_1 t^{-1} + c_2 t^{3/2}$$

* Motivation for the Wronskian

Consider the I.V.P. $\begin{cases} p(t)\,y'' + q(t)\,y' + r(t)\,y = 0 \\ \text{with } y(t_0) = y_0,\ \ y'(t_0) = (y')_0 \end{cases}$

- Form $y(t) = c_1 y_1 + c_2 y_2$ where y_1 & y_2 are solutions
- Apply I.C.

(i) $y(t_0) = c_1 y_1(t_0) + c_2\, y_2(t_0)$

diff't $y'(t) = c_1 y_1'(t) + c_2 y_2'(t)$ @ t_0

(ii) $y'(t_0) = c_1\, y_1'(t_0) + c_2\, y_2'(t_0)$

These result in two eqns & two unknowns.

$$\Rightarrow \quad \begin{aligned} c_1 \overbrace{y_1(t_0)}^{a} + c_2 \overbrace{y_2(t_0)}^{b} &= y_0 \\ c_1 \underbrace{y_1'(t_0)}_{d} + c_2 \underbrace{y_2'(t_0)}_{f} &= y_0' \end{aligned}$$

Via Cramer's Rule: Given $\begin{aligned} ax+by&=c \\ dx+fy&=g \end{aligned}$ then

$$x = \frac{\begin{vmatrix} c & b \\ g & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & f \end{vmatrix}} \quad \text{\&} \quad y = \frac{\begin{vmatrix} a & c \\ d & g \end{vmatrix}}{\begin{vmatrix} a & b \\ d & f \end{vmatrix}}$$

the solution of c_1 & c_2 relies on the

$$\text{determinant} \underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{\text{Wronskian}}$$

not being zero for a solution to exist.

So when the set $\{y_1, y_2\}$ is Lin. Indp we call it a <u>fundamental Solution Set</u> to the ODE

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$\boxed{\text{ex}}$ $\{\sinh(t), e^{-t}, e^{t}\}$ are Lin. <u>Dependent</u>, functions {one function too many}.

but $\{e^{-t}, e^{t}\}$ is <u>Lin Independent</u>

or $\{\sinh(t), e^{t}\}$ <u>Lin. Independent</u> also

i.e. any two is a <u>fundamental set</u>.

$\boxed{Ex}$ In the previous lecture we used

$\underline{y_1 = e^{\lambda t}\cos(t)}$ & $\underline{y_2 = e^{\lambda . t}\sin(t)}$ as though they were Lin. Indep. {Real & Imaginary parts of $e^{(\lambda \pm \mu i)t}$}

prove this via the Wronksian.

$$W = \begin{vmatrix} \overset{y_1}{e^{\lambda t}\cos(\mu t)} & \overset{y_2}{e^{\lambda t}\sin(\mu t)} \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

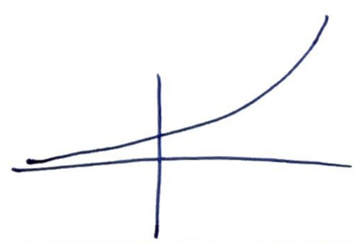
$$= \cancel{e^{\lambda t}\cos(\mu t)\,\lambda e^{\lambda t}\sin(\mu t)} + e^{\lambda t}\cos(\mu t)\,\mu e^{\lambda t}\cos(\mu t)$$

$$- \left[\cancel{e^{\lambda t}\sin(\mu t)\,\lambda e^{\lambda t}\cos(\mu t)} - e^{\lambda t}\sin(\mu t)\,\mu e^{\lambda t}\sin(\mu t)\right]$$

$$= 2\mu e^{\lambda t}\cos^2(\mu t) + 2\mu e^{\lambda t}\sin^2(\mu t)$$

$$= 2\mu e^{\lambda t}\left[\underbrace{\cos^2(\mu t) + \sin^2(\mu t)}_{=1}\right]$$

$$= 2\mu e^{\lambda t} \quad \neq 0$$



So $\{e^{\lambda t}\cos(\mu t),\ e^{\lambda t}\sin(\mu t)\}$ are indeed Lin. Independent

So $y(t) = \underline{c_1 e^{\lambda t}\cos(\mu t)} + \underline{c_2 e^{\lambda t}\sin(\mu t)}$ is the gen. soln.

of the ode $ay'' + by' + cy = 0$ when

$ar^2 + br + c = 0$ yeilds roots $r = \lambda \pm \mu i$

Finally lets study (EX #3) from text

EX #3 text: We know $y_1 = \frac{\sin x}{\sqrt{x}}$ is a soln of

$$x^2 y'' + xy' + (x^2 - \tfrac{1}{4})y = 0 \quad \text{on } (0,\pi)$$

Find a second soln using the formula

- $\div x^2 \Rightarrow \boxed{y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0}$ $\quad P = \frac{1}{x}$

- formula $\quad y_2(x) = \left(\frac{\sin x}{\sqrt{x}}\right) \int \frac{e^{-\int \frac{1}{x}dx}}{\left(\frac{\sin x}{\sqrt{x}}\right)^2} dx$

$$= \frac{\sin x}{\sqrt{x}} \int x \frac{e^{\ln|x^{-1}|}}{\sin^2(x)} dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \frac{x(x^{-1})}{\sin^2(x)} dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \csc^2(x)\, dx$$

$$= \frac{\sin x}{\sqrt{x}} (-\cot x)$$

$$y_2 = -\frac{\cos x}{\sqrt{x}}$$

$$\left\{ y_1 = \frac{\sin x}{\sqrt{x}},\ y_2 = \frac{\cos x}{\sqrt{x}} \right\}$$

these are Lin Ind

$\left\{ \text{by def: } c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} = 0 \Rightarrow c_1, c_2 = 0 \right\}$