Chapter 4: 2nd order ODEs

4.1 Homogeneous Linear 2nd order ODEs with Const. coefficients.

Notice: Homogenous ODE in 1st Order Equs. means we have the form $y' = F\left(y/x\right)$

ex: $y' = \dfrac{xy - y^2}{x^2}$

which can be re-written as

$$y' = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 \quad \text{so indeed } F = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

basically wherever we see y we see an x under it. This is made linear (separable) through the substitution of $u = y/x$ then $y' = u'x + u$

⊛ In 2nd order ODE's we use the word homogeneous to mean that there is no "driving term" on the RHS of the ODE.

EX $t^2y'' - ty' + y = t^2 - \sin(t)$ Non-homogeneous

but $t^2y'' - ty' + y = 0$ is homogeneous

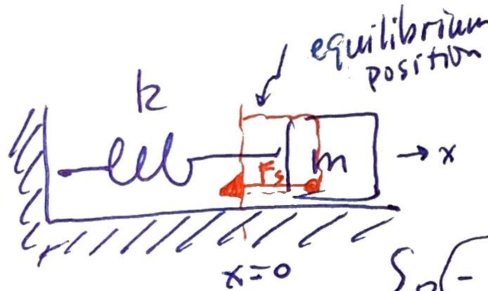
$$p(t)\,y'' + q(t)\,y' + r(t)\,y = \underbrace{g(t)}_{\text{driving force}}$$

but we use $p(t) = \text{constant}$, $q(t) = \text{constant}$
and $r(t) = \text{constant}$ with $g(t) = 0$

$$\Rightarrow \boxed{ay'' + by' + cy = 0}$$

⊗ Spring - mass

(i) No friction, nor driving forces



$\sum F = ma$ {Newton's Law

$F_{sp} = -kx$ {Hooke's Law

So $\boxed{-kx = ma}$

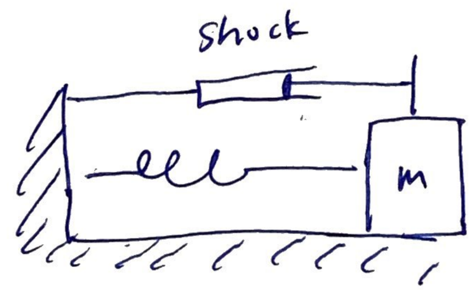
but $a = \frac{dv}{dt}$, $v = \frac{dx}{dt}$ $\Rightarrow$ $a = \frac{d^2x}{dt^2}$

Then Newton's Eqn becomes:

$$-kx = m\frac{d^2x}{dt^2}$$

$$\Rightarrow mx'' + kx = 0 \Rightarrow \boxed{x'' + \left(\frac{k}{m}\right)x = 0}$$

(ii) Add friction



$F_{sp} = -kx$

$F_{drag} = -bv$

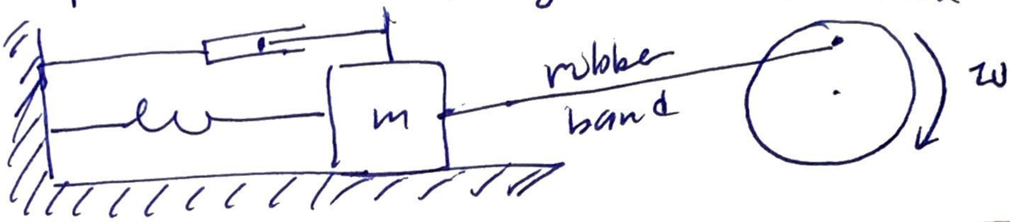
$\boxed{-kx - bv = ma}$ Newtons Law

$b = $ drag coefficient

$$m\underbrace{x''}_{a} + b\underbrace{x'}_{v} + kx = 0$$

$$\boxed{x'' + \left(\frac{b}{m}\right)x' + \left(\frac{k}{m}\right)x = 0}$$ Damped spring mass system

(iii) Damped Driven spring mass: add a driving force — Turntable



$$\boxed{x'' + \left(\frac{b}{m}\right) x' + \left(\frac{k}{m}\right) x = A\cos(\omega t + \phi)}$$

non-homog. 2nd order ODE w/ const. coefficients

Solutions to these ODEs.

- Recall 1st order ODE:

ex: $y' - 9y = 0$

$$\frac{dy}{y} = 9\,dt \Rightarrow \boxed{y = Ce^{3t}}$$

- for a 2nd order, consider an ODE like...

$$y'' - 9y = 0$$

Recall also that $\dfrac{de^t}{dx} = e^t$

So together lets try a solution form that is $\boxed{y = c_1 e^{rt}}$,

then $y' = c_1 r e^{rt}$, $y'' = c_1 r^2 e^{rt}$

and the ODE, $y'' - 9y = 0$ becomes

$$c_1 r^2 e^{rt} - 9(c_1 e^{rt}) = 0$$

$$\Rightarrow c_1 e^{rt}(r^2-9)=0$$

but $e^{rt} \neq 0$ so $\div c_1$ & e^{rt}

$$\Rightarrow \boxed{r^2-9=0 \Rightarrow r=\pm 3}$$

so $\underline{\underline{y(t)=c_1 e^{3t}}}$ or $\underline{\underline{y(t)=c_2 e^{-3t}}}$

* It turns out that the general soln to $\boxed{y''-9y=0}$ is a linear combination of the two:

$$\boxed{y(t)=c_1 e^{3t}+c_2 e^{-3t}}$$

Thm: Superposition:

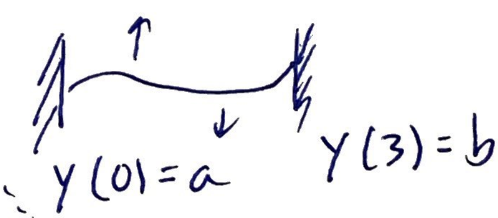
If $y_1(t)$ and $y_2(t)$ are solutions to a linear homogeneous ODE, then so is

$$y(t)=c_1 y_1(t)+c_2 y_2(t)$$

Q: How do we determine c_1 and c_2?

A: we need two conditions. They come in two varieties:

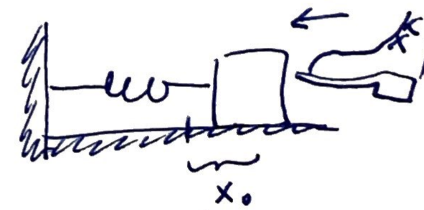
- Boundary Value Problem:



- Initial Values Problems:

$y(0)=a,\quad y'(0)=b$

osition $\quad$ velocity



EX Solve $y'' - 9y = 0$ w/ $\begin{cases} y(0) = 2 \\ y'(0) = -1 \end{cases}$

- we saw that the gen. soln is

$$\underline{y(t) = c_1 e^{-3t} + c_2 e^{3t}}$$

we determine c_1 and c_2 via the two I.Conditions.

1st $\quad \boxed{y'(t) = -3c_1 e^{-3t} + 3c_2 e^{3t}}$

- I.C.
  - Apply $y(0) = 2$: $\quad y(0) = c_1 e^{-3\cdot 0} + c_2 e^{3\cdot 0}$
    $\boxed{2 = c_1 + c_2}$
  - Apply $y'(0) = -1$: $\quad y'(0) = -3c_1 e^{-3\cdot 0} + 3c_2 e^{3\cdot 0}$
    $\boxed{-1 = -3c_1 + 3c_2}$

So 2 eqns with 2 unknowns:

$$\begin{cases} c_1 + c_2 = 2 \\ 3c_1 - 3c_2 = 1 \end{cases}$$

methods of solutions
- graphical
- substitution
- elimination
- Gaussian-Jordan
- Cramers Rule
- matrix inversion
  - armani
  - cofactor
  - 2x2 method

- Solve:

Lets use matrix inversion:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Soln:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbb{A}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

we need $\mathbb{A}^{-1}$:

For $\mathbb{A}^{-1}$ we use the "Armani Method"

$$\left[\mathbb{A} \mid \mathbb{I}\right] \xrightarrow[\text{Row ops}]{\text{Elem.}} \left[\mathbb{I} \mid \mathbb{A}^{-1}\right]$$

Here goes

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} * -3 \\ \leftarrow \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 \end{array}\right] * 6$$

$$\left[\begin{array}{cc|cc} 6 & 6 & 6 & 0 \\ 0 & -6 & -3 & 1 \end{array}\right] +$$

$$\left[\begin{array}{cc|cc} 6 & 0 & 3 & 1 \\ 0 & -6 & -3 & 1 \end{array}\right] \begin{array}{l} \div 6 \\ \div -6 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/6 \\ 0 & 1 & 1/2 & -1/6 \end{array}\right]$$

$$\boxed{\mathbb{A}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}}$$

Find $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1/6 \\ 1/2 & -1/6 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{aligned} c_1 &= \tfrac{1}{2} \cdot 2 + \tfrac{1}{6} \cdot 1 \\ c_2 &= \tfrac{1}{2} \cdot 2 - \tfrac{1}{6} \cdot 1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= 7/6 \\ c_2 &= 5/6 \end{aligned}$$

- Specific Soln:

$$\boxed{y(t) = \frac{7}{6} e^{-3t} + \frac{5}{6} e^{3t}}$$ Solution

to $y'' - 9y = 0 \quad y(0) = 2, \; y'(0) = -1$

⊛ The formallities, for liner **only**!!

For $\boxed{ay'' + by' + cy = 0}$, linear,

- assume the solution is to have the form of:

$$y(t) = e^{rt}$$

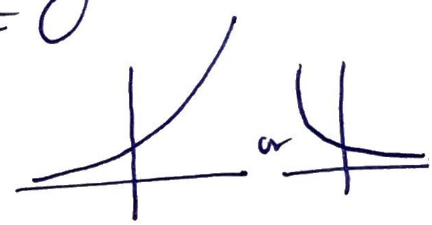
then $\quad y'(t) = re^{rt}$

and $\quad y''(t) = r^2 e^{rt}$

- Substitute these into the ODE to get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$\div e^{rt}$ (which is never zero)



$$\boxed{ar^2 + br + c = 0}$$

This is the characteristic eqn for the ODE.

- This is a quadratic and yeilds, generally, two solutions, r_1 and r_2, say.

we get two solutions to the ODE:

$$\boxed{y_1(t) = C_1 e^{r_1 t} \quad \text{and} \quad y_2(t) = C_2 e^{r_2 t}}$$

Note the cases of : r_1 & r_2 can be real and distinct, and $r_1 = r_2$ double roots
r_1 & r_2 can be complex conjugates

Ex Solve $y'' + 11y' + 24y = 0$, $y(0) = 0$, $y'(0) = -7$ 8

(i) characteristic eqn:

$$r^2 + 11r + 24 = 0$$

$$(r+8)(r+3) = 0 \Rightarrow r_1 = -8,\ r_2 = -3$$

(ii) gen soln: $\boxed{y(t) = c_1 e^{-8t} + c_2 e^{-3t}}$

(iii) IVP:

$y(0) = 0 \Rightarrow \boxed{c_1 + c_2 = 0}$

2 eqns
2 unk.

so $\begin{cases} y'(t) = -8c_1 e^{-8t} - 3c_2 e^{-3t} \\ y'(0) = -8c_1 - 3c_2 \\ -7 = \end{cases} \Rightarrow \boxed{-8c_1 - 3c_2 = -7}$

• Solve $\boxed{\begin{aligned} c_1 + c_2 &= 0 \\ -8c_1 - 3c_2 &= -7 \end{aligned}}$ $c_1 = -c_2$ $\Rightarrow \boxed{c_1 = \frac{7}{5}}$

$-8(-c_2) - 3(c_2) = -7 \Rightarrow \boxed{c_2 = -\frac{7}{5}}$

(iv) specific Solution:

$$\boxed{y(t) = \frac{7}{5} e^{-8t} - \frac{7}{5} e^{-3t}}$$

$\boxed{\text{EX}}$ Outline of a "not so pretty" problem

$$y'' - 6y' - 2y = 0$$

$$\Rightarrow \quad r^2 - 6r - 2 = 0$$

$$\Rightarrow \quad r_{1,2} = 3 \pm \sqrt{11}$$ real radical conjugates

$$r_1 = 3 + \sqrt{11}, \qquad r_2 = 3 - \sqrt{11}$$

$$\boxed{y(t) = C_1 e^{(3+\sqrt{11})t} + C_2 e^{(3-\sqrt{11})t}}$$ general soln. of the o.d.e.