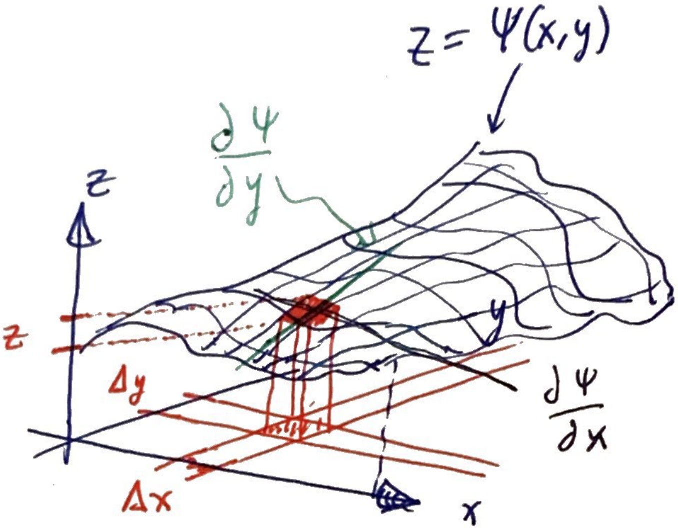
Zill s2

$\boxed{2.4a}$ Exact Equations

skip pt "b"

The linear ODE with continuous coeff & driving funct will not if we have some non linear Behavior. So we examine some different approaches.

* The total derivative



$$\boxed{d\Psi(x,y) = \overset{M}{\frac{\partial \Psi}{\partial x}} \cdot dx + \overset{N}{\frac{\partial \Psi}{\partial y}} \cdot dy} = \vec{\nabla}\Psi \cdot d\vec{x}$$

$\dfrac{\partial \Psi(x,y)}{\partial x}$ = hold 'y' as a constant and diff't w.r.t. "x"

let $\Psi = y^2 + (x^2+1)y - 3x^3$, for example

$$\frac{\partial \Psi}{\partial x} = 0 + 2xy - 9x^2 \quad \leftarrow y = \text{const.}$$

$$\frac{\partial \Psi}{\partial y} = \underline{2y + (x^2+1) - 0} \quad \leftarrow x = \text{const.}$$

So here

$$\underline{\underline{d\Psi = (2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy}}$$

BTW: @ $(x,y) = (3,4)$, if $\Delta x = 0.05$ & $\Delta y = 0.1$ how much does Ψ change?

ow, We seek a ψ such that

$$d\,\psi(x, y(x)) = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy$$

matches our ODE (which we will write as $M(x,y)\,dx + N(x,y)\,dx = 0$)

{This will require that $\psi(x,y) = $ constant so that $d\psi = 0$ }

$\boxed{\text{EX}}$ $$\underbrace{(2xy - 9x^2)}_{M}\,dx + \underbrace{(2y + x^2 + 1)}_{N}\,dy = 0$$

Find ψ such that

$$\frac{\partial \psi}{\partial x} = M \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N$$

Here

$$\frac{\partial \psi}{\partial x} = 2xy - 9x^2, \quad \frac{\partial \psi}{\partial y} = 2y + x^2 + 1$$

Q: what is ψ?

We will discover $\boxed{\psi = x^2y - 3x^3 + y^2 + y + k}$

these requirements are met. } $\frac{\partial \psi}{\partial x} = 2xy - 9x^2$, $\frac{\partial \psi}{\partial y} = x^2 + 2y + 1$

$\frac{\partial^2 \psi}{\partial y \partial x} = 2x$ $\quad$ $\frac{\partial^2 \psi}{\partial x \partial y} = 2x$

Thus The ODE then, is a total derivative of ψ:

$$d\psi = (2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0 \text{ so } \boxed{\psi = \text{const}}.$$

this makes $\boxed{x^2y - 3x^3 + y^2 + y + k = 0}$ Thus

1

(2.4 cont.)

$$\overbrace{(4x^3y^3+3x^2)}^{M=\partial\Psi/\partial x}\,dx + \overbrace{(3x^4y^2+6y^2)}^{N=\partial\Psi/\partial y}\,dy = 0$$

EX Solve $(4x^3y^3+3x^2)dx + (3x^4y^2+6y^2)dy=0$

(i) Is the ODE in gen. form? Yes. $M\,dx+N\,dy=0$

(ii) Is this ODE exact? $\partial M/\partial y \overset{?}{=} \partial N/\partial x \iff \Psi_{xy} \overset{?}{=} \Psi_{yx}$

$$\frac{\partial(4x^3y^3+3x^2)}{\partial y} \overset{?}{=} \frac{\partial(3x^4y^2+6y^2)}{\partial x}$$

$$4x^3\cdot 3y^2+0 \overset{?}{=} 3\cdot 4x^3y^2+0$$ Yes! ✓ can proceed.

Find easier approach

(iii) $\Psi=\int M\,dx$ or $\Psi=\int N\,dy$, which is easier? same level of difficulty.

Build Ψ:

(iv) $\Psi(x,y)=\int(4x^3y^3+3x^2)dx+h(y)$

$$\boxed{\Psi = 4y^3\left(\frac{x^4}{4}\right)+3\left(\frac{x^3}{3}\right)+h(y)} = \underline{y^3x^4+x^3+h(y)}$$

(v) Find $h(y)$: use $\frac{\partial\Psi}{\partial y}=N$

$$\frac{\partial(y^3x^4+x^3+h(y))}{\partial y}=3x^4y^2+6y^2$$

$$\cancel{3y^2x^4}+0+h'(y)=\cancel{3x^4y^2}+6y^2$$

$$h'(y)=6y^2$$

integrate $h(y)=2y^3+C$

$$\boxed{\Psi(x,y)=y^3x^4+x^3+(2y^3+c)}$$

(vi) The solution has $\Psi=$ Const.

$$\Rightarrow \boxed{y^3x^4+x^3+2y^3=C}$$ implicit soln.

we can solve for $y(x)=\sqrt[3]{\dfrac{C-x^3}{2+x^4}}$ This is the soln of $(4x^3y^3+3x^2)dx+(3x^4y^2+6y^2)$

Solve $(2x-2y^2)\,dx + (12y^2-4xy)\,dy = 0$

(i) Form? yes

(ii) Exact? $\dfrac{\partial(2x-2y^2)}{\partial y} \overset{?}{=} \dfrac{\partial(12y^2-4xy)}{\partial x}$

$$-4y \overset{?}{=} -4y \quad \text{yes! proceed.}$$

(iii) Choice? $\underbrace{\int \overset{M}{(2x-2y^2)}\,dx}_{\text{easier}}$ or $\int \overset{N}{(12y^2-4xy)}\,dy$?

(iv) Build ψ: $\psi = \int (2x-2y^2)\,dx + h(y)$

$$\psi = \frac{2x^2}{2} - 2y^2x + h(y)$$

$$\psi = x^2 - 2y^2x + h(y)$$

(v) Find $h(y)$: $\dfrac{\partial \psi}{\partial y} = N$ $\qquad$ $\dfrac{\partial \psi}{\partial x} = M$

$$\frac{\partial(x^2-2y^2x+h(y))}{\partial y} = 12y^2-4xy$$

$$0 - \cancel{4yx} + h'(y) = 12y^2 - \cancel{4xy}$$

$$h'(y) = 12y^2 \Rightarrow \underline{\underline{h(y) = 4y^3 + c}}$$

$$\psi(x,y) = x^2 - 2y^2x + 4y^3 + c$$

(vi) The soln is $\psi = \text{Const.}$

$$\Rightarrow \boxed{x^2 - 2y^2x + 4y^3 = C} \quad \text{implicit Soln.}$$

Solve $(ye^{xy}\tan x + e^{xy}\sec^2 x)dx + (xe^{xy}\tan x)dy = 0$

(i) Form? yes

(ii) Exact? $\dfrac{\partial(ye^{xy}\tan x + e^{xy}\sec^2 x)}{\partial y} \stackrel{?}{=} \dfrac{\partial(xe^{xy}\tan x)}{\partial x}$

$\tan x\dfrac{\partial(ye^{xy})}{\partial y} + \sec^2 x\dfrac{\partial(e^{xy})}{\partial y} \stackrel{?}{=} \cancel{1\cdot e^{xy}\tan x} + xye^{xy}\tan x + \cancel{xe^{xy}\sec^2 x}$

$\tan x(\cancel{1\cdot e^{xy}} + yxe^{xy}) + \cancel{(\sec^2 x)xe^{xy}} \stackrel{?}{=}$

EXACT!

(iii) $\int M dx$ or $\int N dy$? easier

(iv) Build ψ $\psi = \int N dy$

$\psi = \int xe^{xy}\tan x\, dy + g(x)$

$\psi = x\tan x \int e^{xy} dy + g(x)$

$\psi = \cancel{x}\tan x \dfrac{e^{xy}}{\cancel{x}} + g(x)$

$\boxed{\psi = (\tan x)e^{xy} + g(x)}$

(v) find $g(x)$, use $\dfrac{\partial \psi}{\partial x} = M$

$\dfrac{\partial(e^{xy}\tan x + g(x))}{\partial x} = \overbrace{ye^{xy}\tan x + e^{xy}\sec^2 x}^{M}$

$\cancel{ye^{xy}\tan x} + \cancel{e^{xy}\sec^2 x} + g'(x) = \cancel{ye^{xy}\tan x} + \cancel{e^{xy}\sec^2 x}$

$g'(x) = 0 \Rightarrow \underline{g(x) = C}$

$\boxed{\psi(x,y) = e^{xy}\tan x + C}$

(vi) Set ψ = const. $\boxed{e^{xy}\tan x = C}$ Solve $\underline{y = \frac{1}{x}\ln(c\cdot\cot(x))}$

imp soln

Regarding the last example ...

we have equilibrium solutions

$\boxed{x = n\pi}$ solves the ODE!

$$(\underbrace{ye^{xy}\tan x}_{\frac{\sin(n\pi)}{\cos(n\pi)}=0} + e^{xy}\underbrace{\sec^2 x}_{\frac{1}{\cos n\pi}=\pm 1})\,dx + (\underbrace{xe^{xy}\tan x}_{0})\,dy = 0$$

if $x = n\pi$
$dx = 0$

- $x = \text{const}: dx = 0$
  so $xe^{xy}\underbrace{\tan x}_{0} = 0 \Rightarrow \underline{\underline{x = n\pi}}$

- $y = \text{const}: dy = 0$

what if $y =$ const. ?

we want $y\cancel{e^{xy}}\tan x + \cancel{e^{xy}}\sec^2 x = 0$

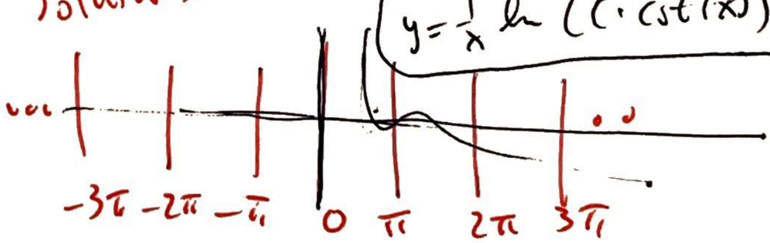
$$\Rightarrow \quad y\frac{\sin x}{\cos x} + \frac{1}{\cos^2 x} = 0$$

$$y = -\frac{1}{\sin x \cos x}$$ but this is not

const. so there is no $\underline{\underline{y = \text{const}}}$ solns

Solutions :

$\boxed{y = \frac{1}{x}\ln\left(C\cdot\csc(x)\right)}$



Solve $(3x+2y^2)\,dx + (2xy)\,dy = 0$

i) Form? yes.

(ii) Exact? $\partial(3x+2y^2)/\partial y \overset{?}{=} \dfrac{\partial\, 2xy}{\partial x}$

$$4y \overset{?}{=} 2y$$

Not Exact.

* Note: if we multiply everything by "x" then the ODE becomes

$$(3x^2+2y^2x)\,dx + (2x^2y)\,dy = 0$$

(ii) Exact? $\dfrac{\partial(3x^2+2y^2x)}{\partial y} \overset{?}{=} \dfrac{\partial(2x^2y)}{\partial x}$

$$4yx \overset{?}{=} 4xy \quad \checkmark$$

exact.

We call x an integration factor!

We skip this extra proceedure, just know that non-exactness is not necc'y the end of the road. Under certain circumstances an integrating ~~factor~~ fa can be found.

We skip integrating factors for Exact eqns. (2.4 part II)

l ut un ce circumstances non-exact can be made into exact eqns. See attahed a .

§2.6 Special Integrating Factors

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, there may exist an integrating factor $\mu = \mu(x,y)$ such that the following equation:

$$\boxed{\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0}$$

is exact.

<u>How do we find the integrating factor</u>? For the new DE to be exact we need

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\Leftrightarrow \quad \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\Leftrightarrow \quad \boxed{\mu(M_y - N_x) = \mu_x N - \mu_y M}$$

This partial DE may have more than one solution and any of these may be used as an integrating factor.

Case 1: $\mu = \mu(x)$. If the integrating factor is a function of x, then $\mu_y = 0$ and the partial derivative becomes total, that is, $\mu_x = \mu'_{(x)}$.

$$\mu(M_y - N_x) = \frac{d\mu}{dx}N \;\Rightarrow\; \frac{d\mu}{\mu} = \frac{M_y - N_x}{N}dx \;\Rightarrow\; \mu_{(x)} = e^{\int \frac{M_y - N_x}{N}dx}$$

Function of x only

Case 2: $\mu = \mu(y)$. If the integrating factor is a function of y, then $\mu_x = 0$ and the partial derivative becomes total, so $\mu_y = \mu'_{(y)}$.

$$\mu(M_y - N_x) = -\frac{d\mu}{dx}M \;\Rightarrow\; \frac{d\mu}{\mu} = -\frac{M_y - N_x}{M}dy \;\Rightarrow\; \mu_{(y)} = e^{-\int \frac{M_y - N_x}{M}dy}$$

Function of y only

Some special cases when we can find μ via a standardized approach:

1) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$ is a function of x only and continuous:

$$\boxed{\mu(x) \;=\; e^{\int g(x)dx}}$$

2) If $-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = h(y)$ is a function of y only and continuous:

$$\boxed{\mu(y) \;=\; e^{\int h(y)dy}}$$

3) We may also seek the integrating factor as $\mu = P(x)Q(y)$. Then,

$$\mu(M_y - N_x) = \mu_x N - \mu_y M$$

$$\Leftrightarrow \quad P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M$$

$$\Leftrightarrow \quad \boxed{M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M}$$

Setting $p(x) = \frac{P'(x)}{P(x)}$ and $q(y) = \frac{Q'(y)}{Q(y)}$, we get:

$$\boxed{M_y - N_x = p(x)N - q(y)M}$$

The functions $p(x)$ and $q(y)$ can be determined by comparing like terms on both sides of the equation. Then,

$$P(x) = \pm e^{\int p(x)dx} \quad \text{and} \quad Q(x) = \pm e^{\int q(x)dx}$$

The sign of a multiplicative integrating factor does not make a difference so we use the simplest possible form for P and Q.

Note: when multiplying the inexact equation we may have lost (or gained) a **constant solution**. Thus, always check if a solution of the form $x \equiv a$ or $y \equiv c$ satisfies the original DE.