2.3 Linear Equations and Integrating Factors

Linear Eqns: y', y are of power 1

General form: $$\boxed{\underbrace{\frac{dy}{dt} + p(t)\,y}_{\text{LHS}} = g(t)}$$

the driving function (RHS)

- Requirements: $p(t)$ and $g(t)$ are continuous.

{if non-continuous see Laplace Transform Chapter 7}

Recall: continuous = can't lift your pen when tracing

OVERALL TACTICS

An integrating factor, $\mu(t)$, is a function that allows $y' + py$ to condense into a derivative of the product of the integrating factor and y.

$$\boxed{[\mu \cdot y]' = \mu y' + \mu' y}$$

This is Not the ODE Just product rule

So the ODE $y' + py = g$ (ODE), when multiplied by $\mu(t)$ becomes

$$\mu y' + \mu p y = \mu g$$

μ * ODE

and the LHS condenses to yeild ...

$$[\mu \cdot y]' = \mu g$$

ODE condensed.

then we integrate and solve for y: $\mu y = \int \mu g$

$$y = \frac{1}{\mu}\int \mu g$$ ANS

* Again, but In **detail**, we will derive a formula for this process which works for all Linear ODE's with continuous coefficients and driving function

- Start with the ODE in std. form:

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

- multiply by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \boxed{\underbrace{\mu(t)\,p(t)}_{\mu'(t)}}\,y = \mu(t)\,g(t)$$

assume this reduces to ie $\boxed{\mu p = \mu'}$ a mini ODE by itself.

$$\Rightarrow \quad \mu y' + \mu' y = \mu g$$

- the LHS condense to yeild the ODE

$$[\mu y]' = \mu g$$

- integrate $\displaystyle\int \frac{d(\mu y)}{\cancel{dt}}\,\cancel{dt} = \int \mu g\,dt$

$\int dy = y$

$\int d(\mu y) = \mu y$

$$\Rightarrow \quad \mu y = \int \mu g\,dt + c$$

- solve for y:

$$\boxed{y(t) = \frac{1}{\mu(t)}\int \mu(t)\,g(t)\,dt + \frac{C}{\mu(t)}}$$

But, what about $\mu(t)$?

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$$\frac{dy}{dt} + p(t)\,y = g(t)$$

- multiply by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \boxed{\underbrace{\mu(t)\,p(t)}_{\mu'(t)}}\,y = \mu(t)\,g(t)$$

assume this reduces to $\mu'(t)$ ie $\boxed{\mu p = \mu'}$ a mini ODE by itself.

$$\Rightarrow \quad \mu y' + \mu' y = \mu g$$

- the LHS condense to yeild the ODE

$$[\mu y]' = \mu g$$

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But, what about $\mu(t)$?

Recall that to condense the LHS we required $\mu p = \mu'$

That is to say

$$\boxed{\frac{d\mu}{dt} = \mu p}$$

- We can solve this! Seperate variables!

$$\frac{d\mu}{\mu} = p\,dt$$

- integrate $\int \frac{d\mu}{\mu} = \int p\,dt + \cancel{C}$ { we need not use a const. of integration }

$$\ln|\mu| = \int p\,dt$$

- raise as powers of e $\quad e \qquad e$

$$\boxed{\mu(t) = e^{\int p\,dt}}$$

This establishes the integrating factor.

All Together:

$$\boxed{\begin{array}{l} \text{The solution of } y' + p(t)y = g(t) \\ \text{is } \quad y(t) = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t) g(t)\,dt + \dfrac{c}{\mu(t)} \\ \text{where } \quad \mu(t) = e^{\int p(t)\,dt} \end{array}}$$

Don't memorize this (just the $\mu(t) = e^{\int p(t)\,dt}$

EX Find the solution of $\frac{dv}{dt} = 10 - \frac{1}{5}v$

(i) Write ODE in std. Form:

$$v' + \frac{1}{5}v = 10 \quad , \text{ where } p = \frac{1}{5}$$

(ii) Find $\mu(t)$

$$\underbrace{\mu(t) = e^{\int \left(\frac{1}{5}\right) dt}}$$

$$\boxed{\mu(t) = e^{\frac{1}{5}t}}$$

(iii) multiply and condense

$$e^{t/5}\left(v' + \frac{1}{5}v\right) = e^{t/5} \cdot 10$$

$$\underbrace{e^{t/5}v' + \frac{e^{t/5}v}{5}} = 10e^{t/5}$$

the whole process in one step.

$$\boxed{\left(e^{t/5}v\right)' = 10e^{t/5}}$$

ODE w/ integ. factor

(iv) integrate and use const. of integration:

$$e^{t/5}v = 10\int e^{t/5}\,dt$$

$u = t/5$
$du = dt/5$

$$e^{t/5}v = 10 \cdot 5 \int e^{u}\,du$$

$$e^{t/5}v = 50e^{t/5} + C$$

$* e^{-t/5}$

(v) Solve for v:

$$\boxed{v(t) = 50 + Ce^{-t/5}}$$

use an I.C. to get "C"

$\boxed{\text{EX}}$ Solve the IVP

$$\cos(x)\,y' + \sin(x)\,y = 2\cos^3(x)\sin(x) - 1$$

where $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$ and $x \in [0, \frac{\pi}{2})$

(i) Get form: $\div \cos(x)$

$$\Rightarrow \quad y' + \tan(x)\,y = 2\cos^2(x)\sin(x) - \sec(x)$$

(ii) Find $\mu(x)$

$$\mu(x) = e^{\int \tan(x)\,dx}$$

$$\mu = e^{\ln(\sec(x))}$$

$$\boxed{\mu = \sec(x)}$$

$\int \tan(x)\,dx$
$= -\ln|\cos x|$
$= \ln\left|\frac{1}{\cos(x)}\right|$
$= \ln|\sec(x)|$

(iii) multiply and condense

$$\underbrace{\sec(x)\,y' + \sec(x)\tan(x)\,y} = \sec(x)\,2\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)\,y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

(iv) integrate w/ constant.

$$\sec(x)\,y = \int \underbrace{2\cos(x)\sin(x)}_{\sin(2x)}dx - \int \sec^2(x)\,dx$$

$$\sec(x)\,y = -\tfrac{1}{2}\cos(2x) - \tan(x) + C$$

(v) Solve for y:

$$y(x) = -\frac{1\cos(2x)}{2\sec(x)} - \frac{\tan(x)}{\sec(x)} + \frac{C}{\sec(x)}$$

$$\boxed{y(x) = -\tfrac{1}{2}\cos(2x)\cos(x) - \sin(x) + C\cdot\cos(x)}$$

Next, use the I.C. to determine "C"

Use

$$y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$\Rightarrow y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\underbrace{\cancel{\cos}\left(2\cdot\frac{\pi}{4}\right)}_{0} - \underbrace{\cancel{\sin}\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}} + c\cdot\underbrace{\cancel{\cos}\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2} \quad , \div\sqrt{2}$$

$$3 = -\frac{1}{2} + \frac{c}{2} \quad , * 2$$

$$6 = -1 + c \quad \Rightarrow \quad \boxed{c = 7}$$

Finally

$$\boxed{y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)}$$