2.2 Separable Equations

When we can gather all y variables on the LHS and all x variables on the RHS we say the ODE is a separable eqn.

Ex $\frac{dy}{dx} = 6y^2 x$, $y(1) = \frac{1}{25}$

$$\frac{dy}{y^2} = 6x\,dx \quad \int \text{integrate RHS}$$

$$y^{-2}dy = 6\frac{x^2}{2} + C$$

$$\frac{y^{-2+1}}{-2+1} = 3x^2 + C$$

$$\frac{-1}{y} = 3x^2 + C$$

apply I.C. $y(1) = \frac{1}{25}$

$$\frac{-1}{(1/25)} = 3(1)^2 + C$$

$$-25 = 3 + C$$

$$-28 = C$$

Final eqn:

$$-\frac{1}{y} = 3x^2 - 28$$

$$\boxed{y = \frac{-1}{3x^2 - 28}}$$

$$x \quad \frac{dy}{dx} = \frac{x}{y}$$

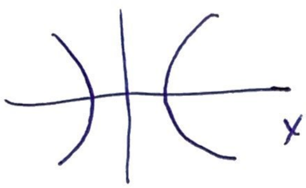
$$y\,dy = x\,dx$$

$$y^2 = x^2 + C$$

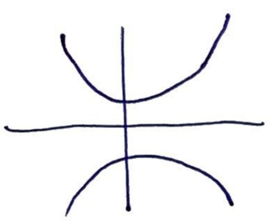
$$\boxed{x^2 - y^2 = C}$$

hyperbolas

• If $C > 0$



• If $C < 0$



• If $C = 0$

$$y^2 = x^2$$

$$\boxed{y = \pm x}$$

assymptotes

I.C.

$y(1) = 1$ "•"

$$1^2 - 1^2 = C$$

$$\Rightarrow C = 0$$

$$x^2 - y^2 = 0$$

"•" $y(-1) = 1$

$y = x$

• Slope field

$$y' = m = \frac{x}{y}$$

Lets try some "m" values

• $m = 1$

$$\boxed{y = x}$$

(slope of isocline $= 1$ also)

• $m = -1$

$$\boxed{y = -x}$$

• $m = 0$

$$0 = \frac{x}{y}$$

$$\boxed{x = 0}$$

• $m = \text{undef}$

$$\infty = \frac{x}{y}$$

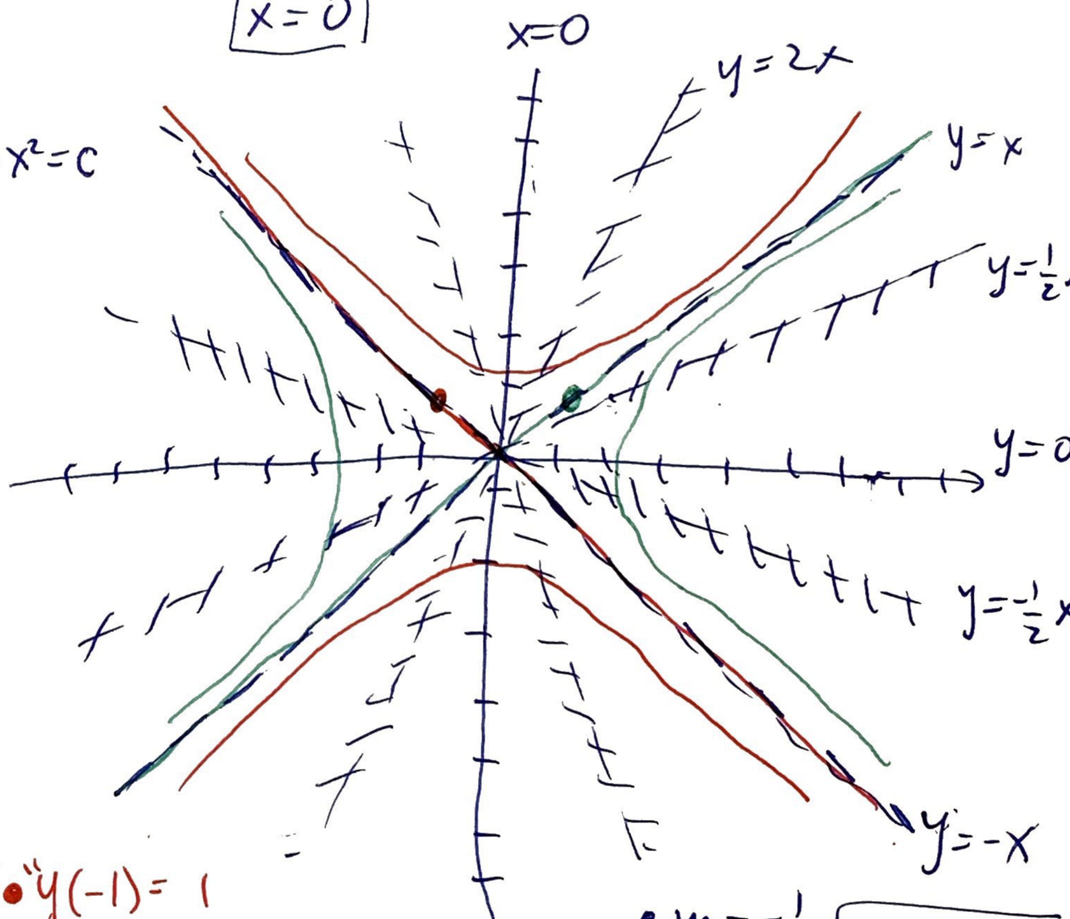
$\boxed{y = 0}$ x-axis $m = \infty$

• $m = 2$

$2 = \frac{x}{y} \Rightarrow \boxed{y = \frac{1}{2}x}$

• $m = -2$ $\boxed{y = -\frac{1}{2}x}$

• $m = 1/2 \rightarrow \boxed{y = 2x}$



• $m = -\frac{1}{2}$ $\boxed{y = -2x}$

ex Solve $y' = y^2 - 4$

$$\Rightarrow \quad \frac{dy}{y^2-4} = dx$$

lets decompose $\frac{1}{y^2-4} = \frac{1}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$

$$\frac{1}{y^2-4} = \frac{A(y-2) + B(y+2)}{(y+2)(y-2)}$$

equate numerators

$$1 = Ay - 2A + By + 2B$$

matching LHS & RHS powers of y:

$$\begin{cases} y^1: (0 = A + B) * 2 \\ y^0: \; 1 = -2A + 2B \end{cases}$$

$+$

$$1 = 4B \longrightarrow B = 1/4 \longrightarrow A = -\frac{1}{4}$$

so $\boxed{\frac{1}{y^2-4} = \frac{-1/4}{y+2} + \frac{1/4}{y-2}}$

Now integrate the ODE: $\frac{dy}{y^2-4} = dx$

$$-\frac{1}{4}\int \frac{dy}{y+2} + \frac{1}{4}\int \frac{dy}{y-2} = \int dx$$

$$-\frac{1}{4} \ln|y+2| + \frac{1}{4} \ln|y-2| = x + C \quad \leftarrow \text{implicit solution}$$

$$(\qquad\qquad\qquad) * 4$$

$$\underbrace{-\ln|y+2| + \ln|y-2|} = 4x + C$$

$$\ln\left(\frac{|y-2|}{|y+2|}\right) = 4x + C$$

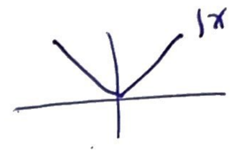
Implicit Soln

• exponentiate

$$\frac{|y-2|}{|y+2|} = Ce^{4x}$$

Q: Is $\frac{|y-2|}{|y+2|} = \left|\frac{y-2}{y+2}\right|$

A: Recall $|x| = \begin{cases} +x & x \geq 0 \\ -x & x < 0 \end{cases}$



$$\frac{+(y-2)}{+(y+2)} \text{ or } \frac{+(y-2)}{-(y+2)} \text{ or } \frac{-(y-2)}{+(y+2)} \text{ or } \frac{-(y-2)}{-(y+2)}$$

$(-)$

$(+)$

$$\Rightarrow \pm\left(\frac{y-2}{y+2}\right) = Ce^{4x} \Rightarrow \frac{y-2}{y+2} = \pm Ce^{4x}$$

• Try to solve for y:

$$y - 2 = \pm(y+2)Ce^{4x}$$

$$y - 2 = \pm yCe^{4x} \pm 2Ce^{4x}$$

$$y \mp Cye^{4x} = 2 \pm 2Ce^{4x}$$

$$y(1 \mp Ce^{4x}) = 2(1 \pm Ce^{4x})$$

$$\boxed{y = 2\frac{(1 \pm Ce^{4x})}{(1 \mp Ce^{4x})}}$$

$\pm$ & C are determined by I.C.

BTW:

note $y' = (y-2)(y+2)$
and if $y = \pm 2$
then $y' = 0$ and RHS $= 0$ also

$\boxed{y = \pm 2}$ is also a soln
we learned in 1.2
we will not have a unique soln in this case.

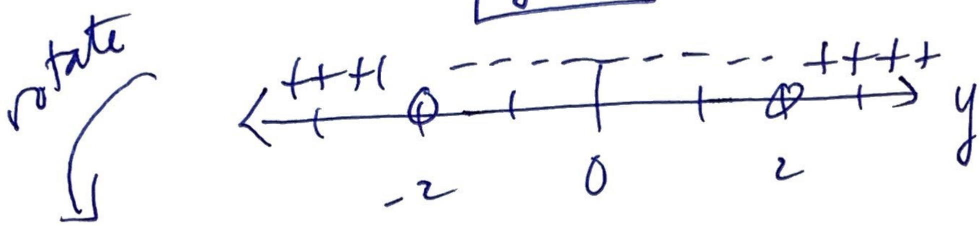
(ex) cont.

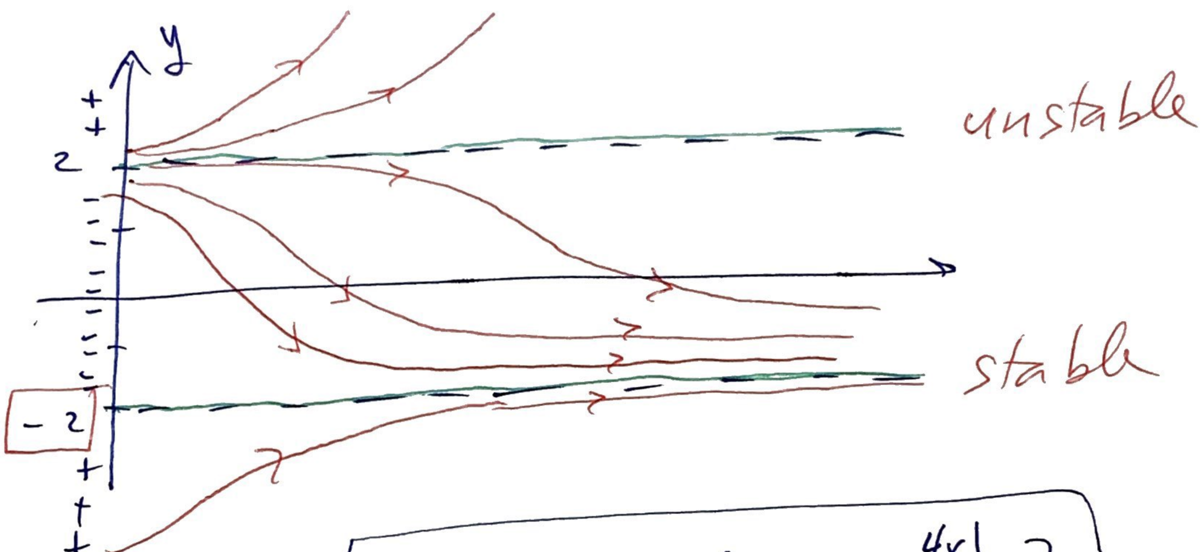
- look at the original problem

ODE: $\boxed{y' = (y-2)(y+2)}$

observe that $\boxed{y=2}$ is a solution (equilibrium)

$\boxed{y=-2}$ is also a soln.





Summary

$$\boxed{\begin{array}{ll} y(x) = 2\dfrac{(1 \pm Ce^{4x})}{(1 \mp Ce^{4x})} & \Big\} \; y(0) \neq 2, -2 \\ y(x) = 2 & y(0) = 2 \\ y(x) = -2 & y(0) = -2 \end{array}}$$

(ex) cont.

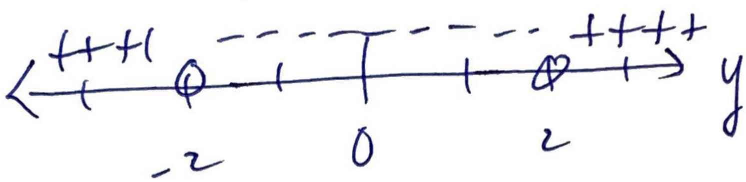
- look at the original problem $\leftarrow$ $y' = F(y)$ only

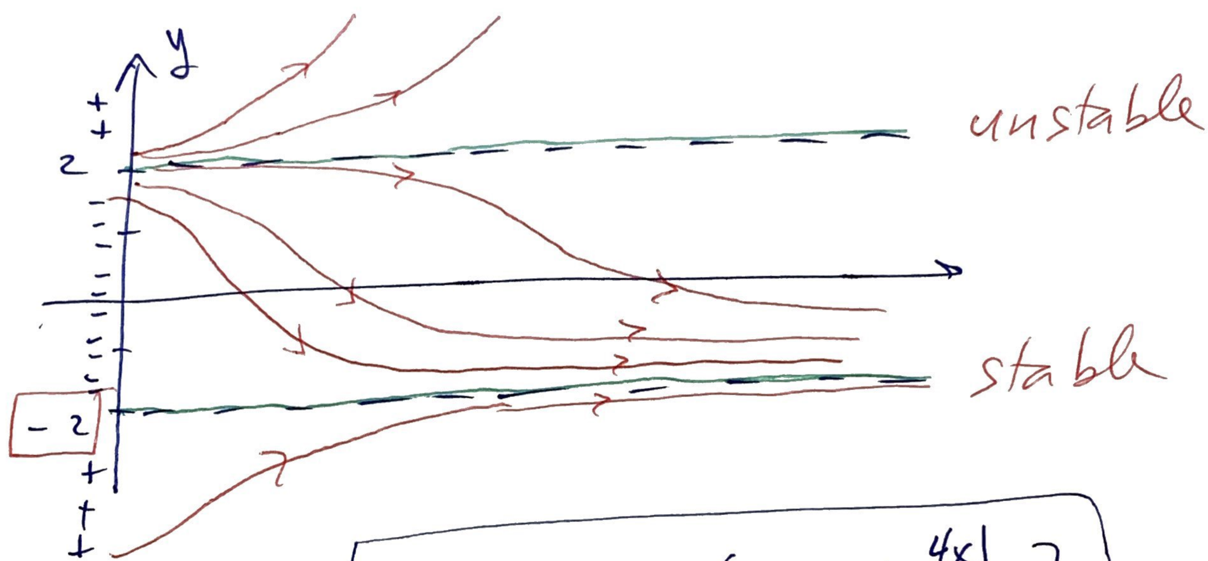
ODE : $\boxed{y' = (y-2)(y+2)}$ an Autonomous Eqn

observe that $\boxed{y=2}$ is a solution (equilibrium)

$\boxed{y=-2}$ is also a soln.

rotate





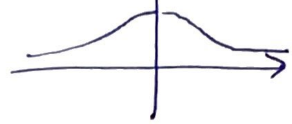
Summary

$$y(x) = 2\,\frac{(1 \pm Ce^{4x})}{(1 \mp Ce^{4x})} \Bigg\} \; \boxed{y(0)} \neq 2, -2$$

$$y(x) = 2 \qquad y(0) = 2$$

$$y(x) = -2 \qquad y(0) = -2$$

$$\frac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$$



$$dy = e^{-x^2}\,dx$$

$$\boxed{y = \int e^{-x^2}\,dx + C}$$ ← normally how we approached I.V.P.

I.C: we can't integrate $\int e^{-x^2}dx$

introduce a parameter "t"

i.e. $y(f(t))$

$y = x^2$

$y = t^2$
$x = t$

$$\int_{t=3}^{t=x} \left(\frac{dy}{dt}\right) dt = \int_{t=3}^{t=x} e^{-t^2}\,dt$$ ← parameter of integration

$$y(t)\Big|_{t=3}^{t=x} = \int_{t=3}^{t=x} e^{-t^2}\,dt$$

Integrate from the I.C. to any x

$$y(x) - \underbrace{y(3)}_{5} = \int_{t=3}^{t=x} e^{-t^2}\,dt$$

$$\boxed{y(x) = 5 + \int_3^x e^{-t^2}\,dt}$$ ← numerically

integral eqn.

- many times we can't evaluate integrals so we treat them as above.
- many times we end up with implicit solns.

$\underline{g(y) = f(x)}$ and cannot isolate $y = F(x)$.