Chapter 2: Solving 1st Order ODE's

2.1 Direction fields and Equilibrium Solutions

- A 1st order linear ODE might have the form $\boxed{p(x)\,y'(x) + q(x)\,y(x) = g(x)}$

- We can write a non-linear ODE as simply

$$\boxed{\frac{dy}{dx} = F(x,y)} \text{ in general}$$

ex $\underbrace{y' = \sqrt{x}}_{F(x)}$, $\underbrace{y' = y^2}_{F(y)}$, $\underbrace{y' = 3x^2y^2 - 14x}_{F(x,y)}$

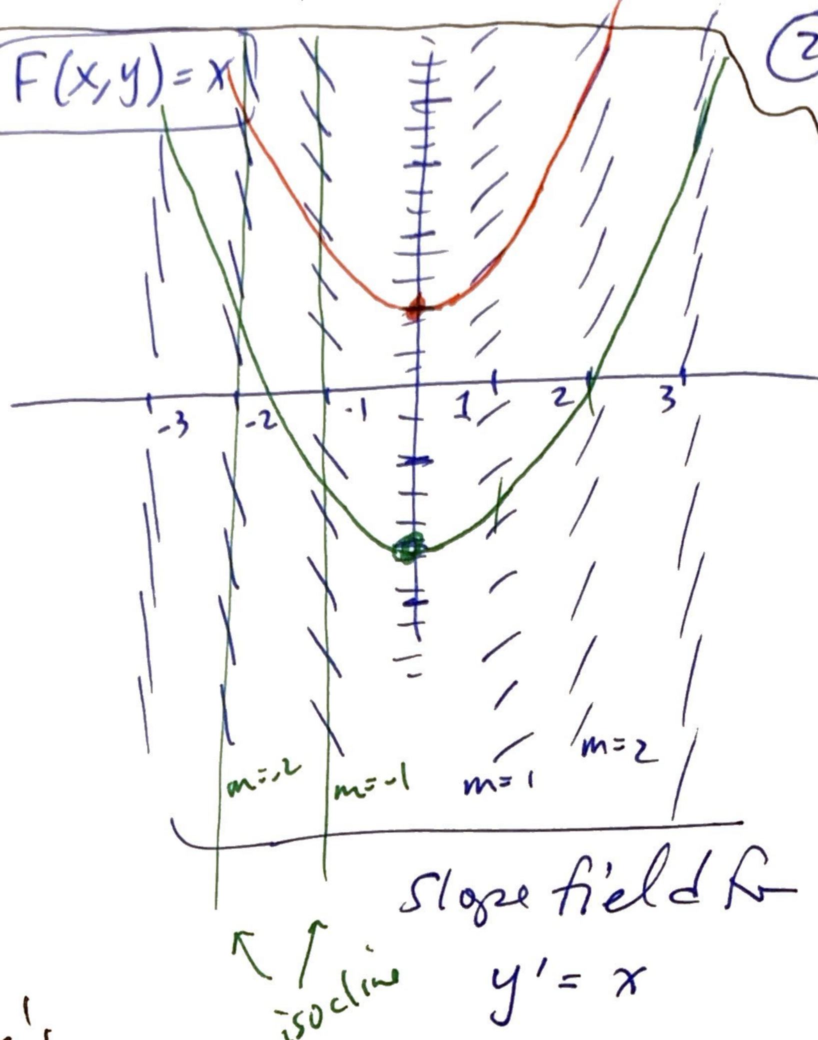
- For 1st order ODE's dy/dx = slope.

- For all (x,y) pairs, therefore, we can insert the x & y values into $F(x,y)$ and know the slope of any trajectory passing through that point.

- The chart of all (x,y) pairs with small line segments tilted according to the slope is called the slope-field for that ODE.

Ex $y' = x$ ← ODE $F(x,y) = x$

x	$m = y' = x$
$x = 0$	$y' = 0$
$x = 1$	$y' = 1$
$x = 2$	$y' = 2$
$x = -1$	$y' = -1$
$x = -2$	$y' = -2$
$x = -3$	$y' = -3$



Slope field for $y' = x$

- trajectories
  $y(0) = 1$
  $y(0) = -2$
- appear to be parabola's.

- exact solution:

$$\frac{dy}{dx} = x$$

$$dy = x\,dx$$

$$\boxed{y = \frac{x^2}{2} + c}$$ Family of solution

- I.C. $y(0) = 1$

$$\boxed{y(x) = \frac{x^2}{2} + 1}$$

trajectory #2.

- I.C. $y(0) = -2$

$$\boxed{y(x) = \frac{x^2}{2} - 2}$$

trajectory #2

specific soln.

Def: An isocline is the curve in a slope-field that has the same slope all along the curve.

$\boxed{\text{Ex}}$ Sketch the slope-field for $y' = \frac{-x}{y}$ ③

$\{$BTW: $yy' = -x \leftarrow$ this is a non-linear ODE$\}$

→ Std. Form $\frac{dy}{dx} = F(x,y)$ here $\boxed{F(x,y) = \frac{-x}{y}}$

m

$\boxed{\text{isoclines}}$

- when $\boxed{m=0}$ $y'=0$

  $m = F$

  $0 = -\frac{x}{y} \Rightarrow \boxed{x=0}$ y-axis

- when $\boxed{m=1}$ $y'=1$

  so $F = -\frac{x}{y} \rightarrow 1 = \frac{-x}{y}$

  $\Rightarrow \boxed{y=-x}$

- $\boxed{m=-1}$

  $-\frac{x}{y} = -1 \rightarrow \boxed{x=y}$

- $\boxed{m = 2}$

  $-\frac{x}{y} = 2 \rightarrow \boxed{y = -\frac{1}{2}x}$

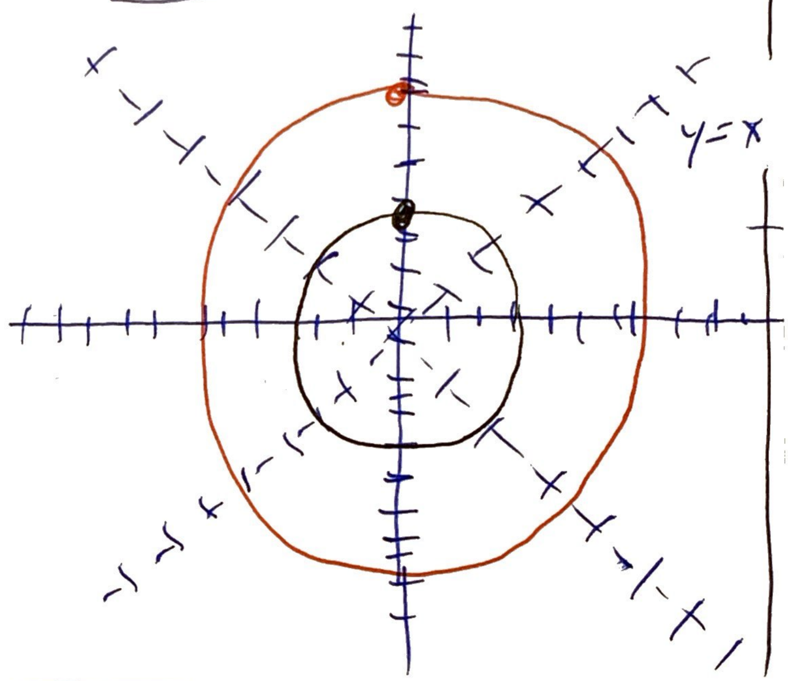
- $\boxed{m = -2}$

  $-\frac{x}{y} = -2 \rightarrow \boxed{y = \frac{1}{2}x}$

$y = 0 \rightarrow m = -\infty$

b/c $m = -\frac{x}{y}$

- $\boxed{\text{sketch}}$



$\boxed{\text{Exact}}$

$\frac{dy}{dx} = -\frac{x}{y} \rightarrow y\,dy = -x\,dx$

- integrate

  $y^2 = -x^2 + c$

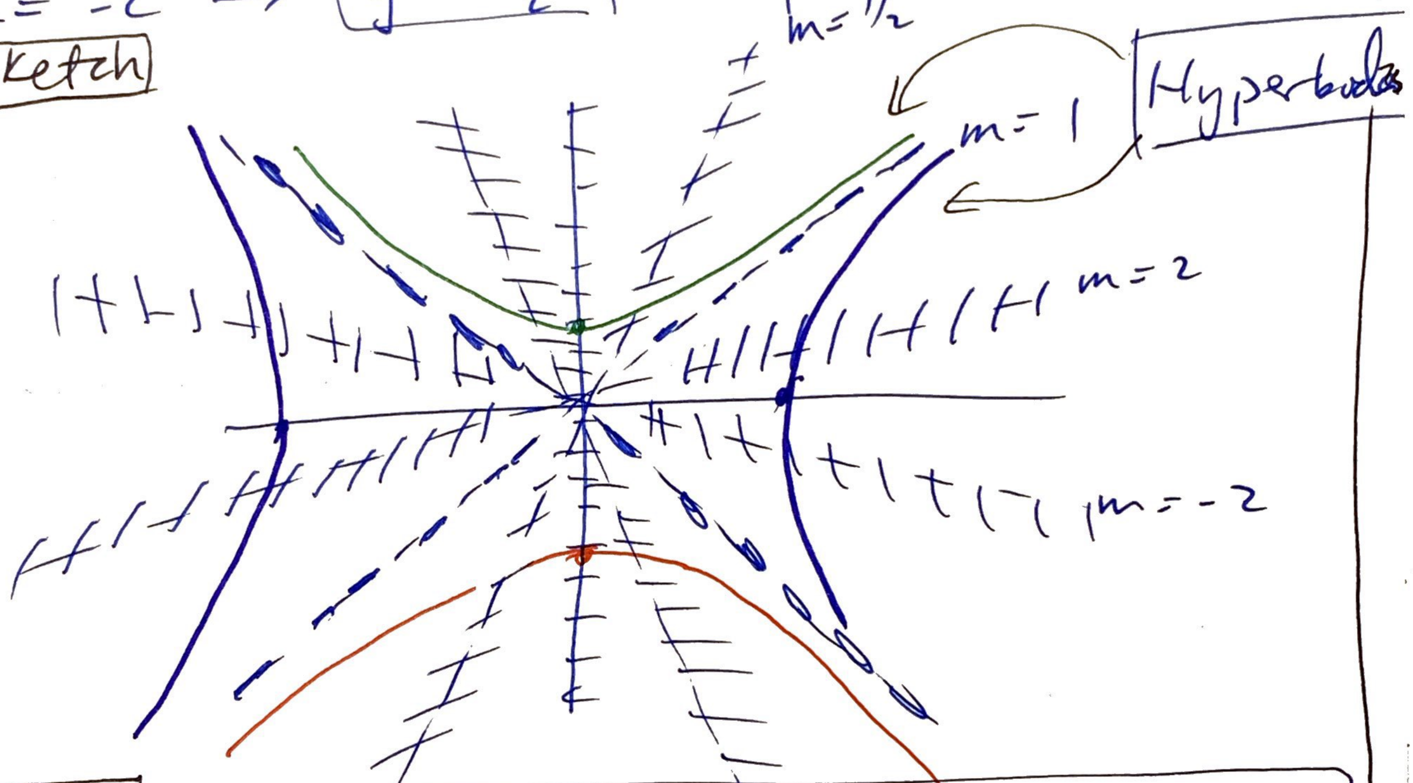
- $\boxed{x^2 + y^2 = c}$ circles

EX Slope-field for $y' = +\frac{x}{y}$

$\boxed{F(x,y) = x/y} \leftarrow m$

- $m=0 \Longrightarrow \frac{x}{y}=0 \quad x=0$ y axis
- $m=1 \Longrightarrow \frac{x}{y}=1$ or $\boxed{y=x}$
- $m=2 \rightarrow \boxed{y=\frac{1}{2}x}$
- $m=-1 \rightarrow \boxed{y=-x}$
- $m=-2 \rightarrow \boxed{y=-\frac{1}{2}x}$
- $m=\frac{1}{2} = \frac{x}{y} \quad \boxed{y=2x}$
- $m=-\frac{1}{2} = \frac{x}{y} \rightarrow y=-2x$

sketch



exact: $\frac{dy}{dx} = \frac{x}{y} \rightarrow y\,dy = x\,dx \rightarrow y^2 = x^2 + C$

So $\boxed{x^2 - y^2 = C}$ families of hyperbola.

* We consider a special type of ODE's.

If the $y' = F(x,y)$ is **F(y) only** we call the ODE's $\boxed{\text{autonomous ODES}}$

Ex Consider $\boxed{\frac{dy}{dx} = y}$ i.e. $F(y) = y$

• Slope-field

slope

isocline

• $\boxed{m=0}$ then $y'=0$ so $\boxed{y=0}$ (x axis)

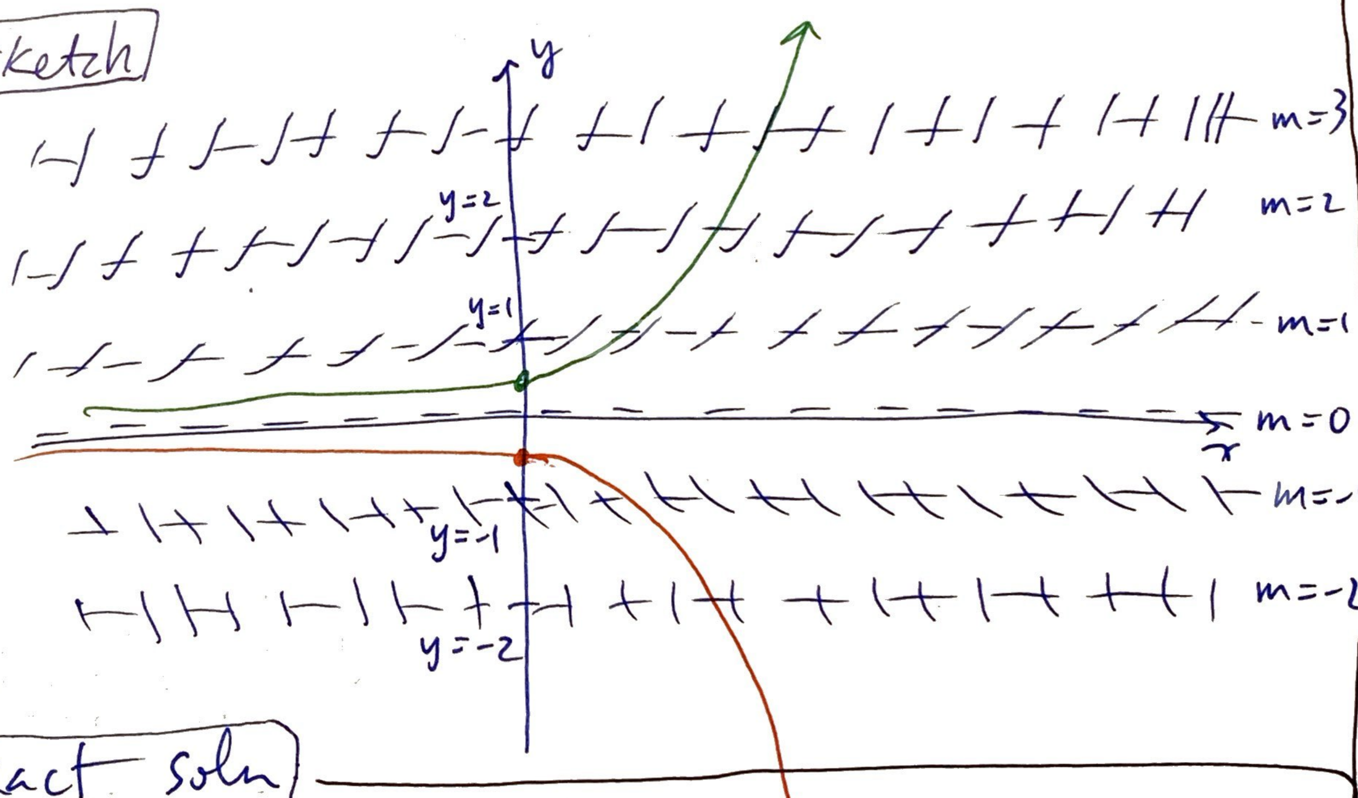
• $\boxed{m=1}$ then $y'=1$ so $\boxed{y=1}$ line

• $m=2$ $\longrightarrow$ $\boxed{y=2}$

• $m=-1$ $\longrightarrow$ $y=-1$

• $m=-2$ $\longrightarrow$ $y=-2$

• sketch



• exact soln

$$\frac{dy}{dx} = y \rightarrow \frac{dy}{y} = dx \rightarrow \ln|y| = x + c \rightarrow \boxed{y = Ce^x}$$

• Note: $m=0$ (ie x-axis) is an asymptote.

Trajectories Do NOT cross asymptotes.

EX Plot the equilibrium solutions for

$$y' = y^2 - y - 6$$

when $y' = 0$ on some line then a solution curve cannot pass through that line

Indeed, as x (or t) increases the soln curve can only approach or diverge from such lines

We classify these "equilibrium solutions" as

stable, if the trajectory approaches the line from either above or below, or unstable, if the trajectories slightly above or below start to move away from the line. semi-stable is a line with both stability on one side & instability on the other side.

- $y' = (y-3)(y+2)$
- $y' = 0$ @ $\boxed{y = 3 \text{ or } -2}$ equilib. solns.
- isoclines

$y = 1$: $y' = (1-3)(1+2) = -6$ so $m = -6$

$y = 2$: $y' = (2-3)(2+2) = -4$ so $m = -4$

$y = 0$: $y' = (0-3)(0+2) = -6$ so $m = -6$

$y = -1$: $y' = (-1-3)(-1+2) = -4$ so $m = -4$

$y = -2$: $y' = (-2-3)(-2+2) = 0$ so $m = 0$

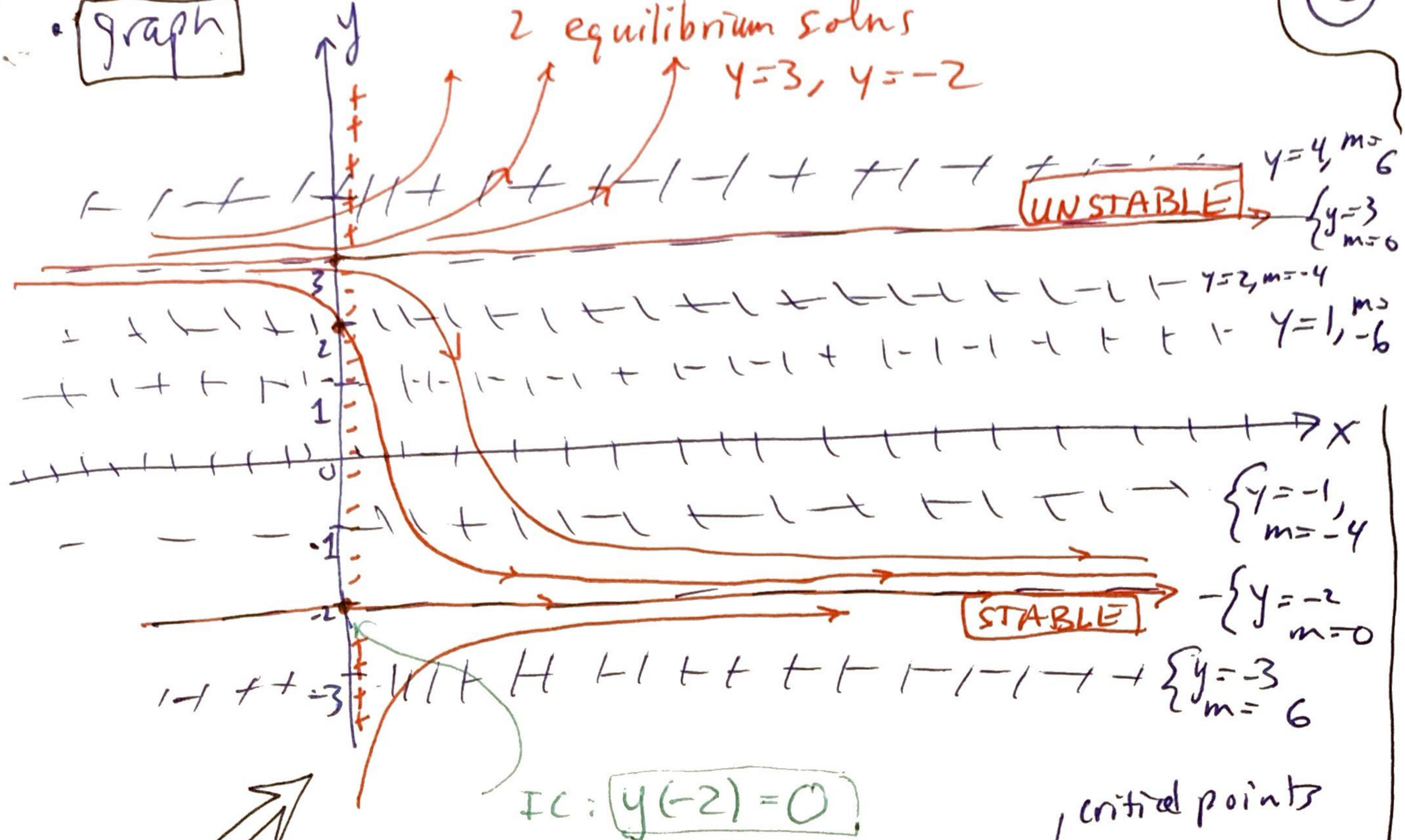
$y = -3$: $y' = (-3-3)(-3+2) = 6$ so $m = 6$

$y = 3$: $y' = (3-3)(3+2) = 0$ so $m = 0$

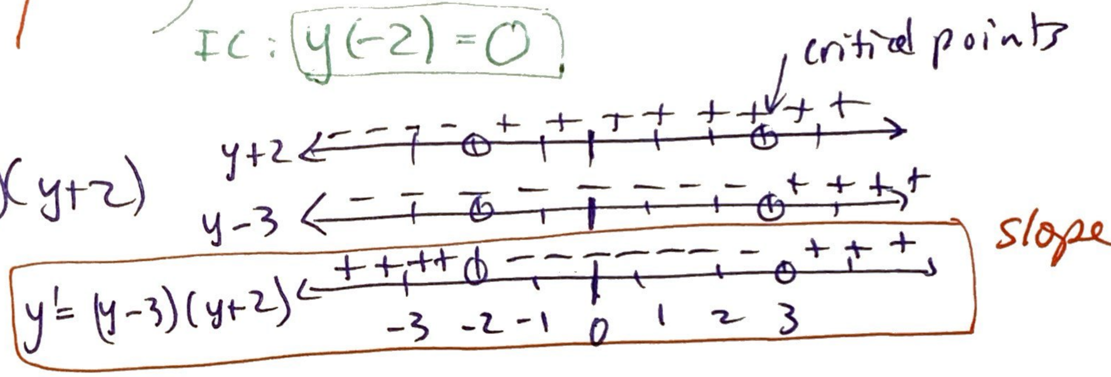
$y = 4$: $y' = (4-3)(4+2) = 6$ so $m = 6$

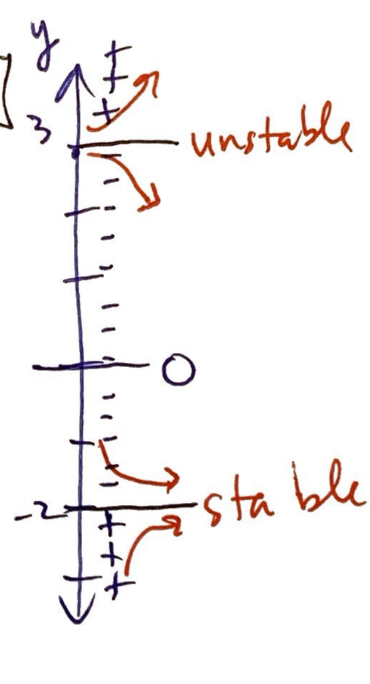
EX: (cont'd)

• graph



• sign analysis (y−3)(y+2)





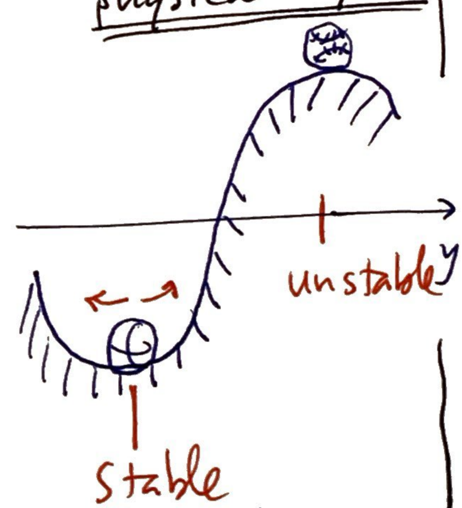
⊕ above

⊖ below

⊖ above

⊕ below

physical equiv.



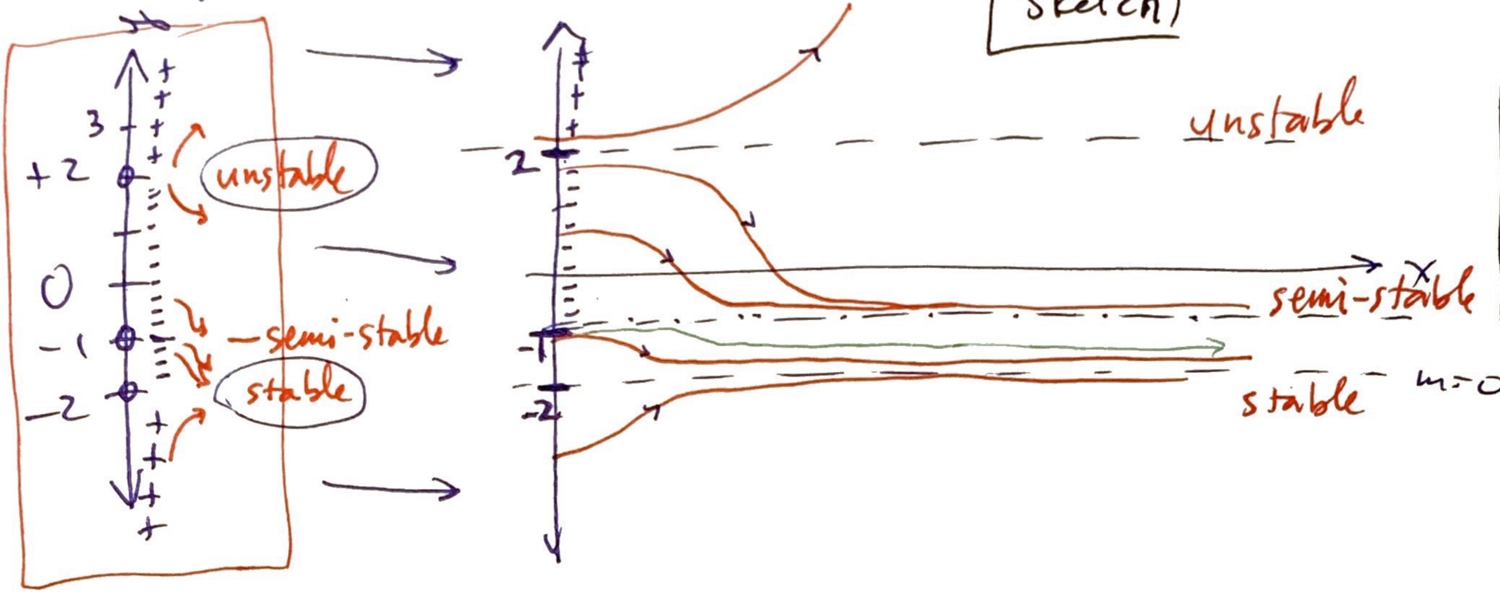
end of example

Ex Classify the equilibrium solutions for

$$y' = (y^2-4)(y+1)^2$$

- factor: $y' = (y+2)(y-2)(y+1)^2$
- eq. solns @ $y = -2, +2, -1, -1$

$\boxed{\text{sketch}}$

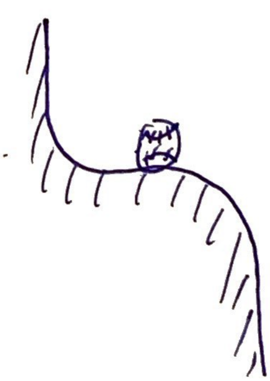


$$\begin{cases} \text{as } x \to \infty & y \to -2, \text{ if } y < -1 \\ \text{as } x \to \infty & y \to -1 \text{ if } y \in (2,-1) \\ \text{as } x \to \infty & y \to \infty \text{ if } y > 2 \end{cases}$$

- $\boxed{\text{analogies:}}$

"unstable"

$y=2$



semi-stable

$y=-1$

"stable"

$y=-2$

- plot on geogebra.com 🔍 slope field

$\boxed{\text{end 2.1}}$