## 1.2 Uniqueness and Existence

①

$Q_1$: when do we know a solution exists to an ?

$Q_2$: If one exists, might there be a second solution.

$\boxed{Q_1 \to \text{existence}}$ $\boxed{Q_2 \to \text{uniqueness}}$

**Thm**: For the 1st order Linear ODE

$$y' + p(t)\,y = g(t) \quad w/ \underbrace{y(t_0) = y_0}_{I.C.}$$

{Non-homog. 1st order, Linear, ODE}

If $p(t)$ and $g(t)$ are continuous on some open intv'l $t \in (\alpha, \beta)$. AND if this intv'l contains the value $t_0$

Then there exists a unique solution in the intv'l

Both in this case.

notes:
- $y_0$ is not mentioned here
- soln exists
- soln is unique (only one)

⊛ Sometimes knowing there exists a solution is just as important as finding the solution. We can use numerical analysis to find the soln even if we could find an analytical solution

⊛ we call the intv'l a "region of validity" or "interval of validity"

$\boxed{\text{EX}}$ Determine the intv'ls of validity for

$$(t^2-9)y' + 2y = \ln|20-4t| \text{ for } y(4) = 13$$

Use the thm for Lin. 1st order ODE's:

- 1st get the form used in the thm: $y' + p(t)y = g(t)$

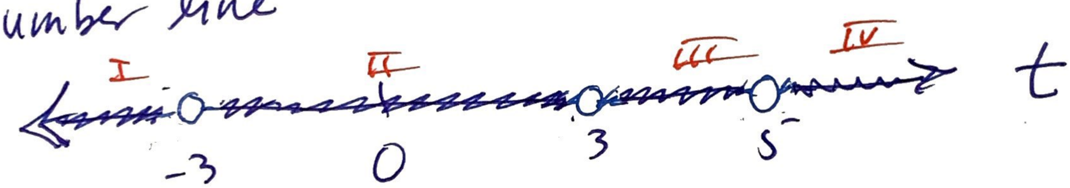
$$\div t^2-9 \Rightarrow \boxed{y' + \underbrace{\left[\frac{2}{(t+3)(t-3)}\right]}_{p(t)} y = \underbrace{\frac{\ln|20-4t|}{(t+3)(t-3)}}_{g(t)}}$$

- Q: where do $p$ & $g$ misbehave?

  $p(t)$ has a problem @ $t = -3, 3$

  $g(t)$ has issues @ $t = -3, 3$ and $t = 5$

- Number line




So any region excluding $-3, +3, +5$ has a unique soln

$\boxed{\{ (-\infty, -3) \text{ or } (-3, 3) \text{ or } (3, 5) \text{ or } (5, \infty) \}}$

BUT for the I.C. $t_0 = 4$ only $(3, 5)$ is gauranteed to have a unique solution since this region contains the I.C.

So the intv'l of validity for this Init. Value Prob. is $\boxed{3 < t < 5}$ ANS:

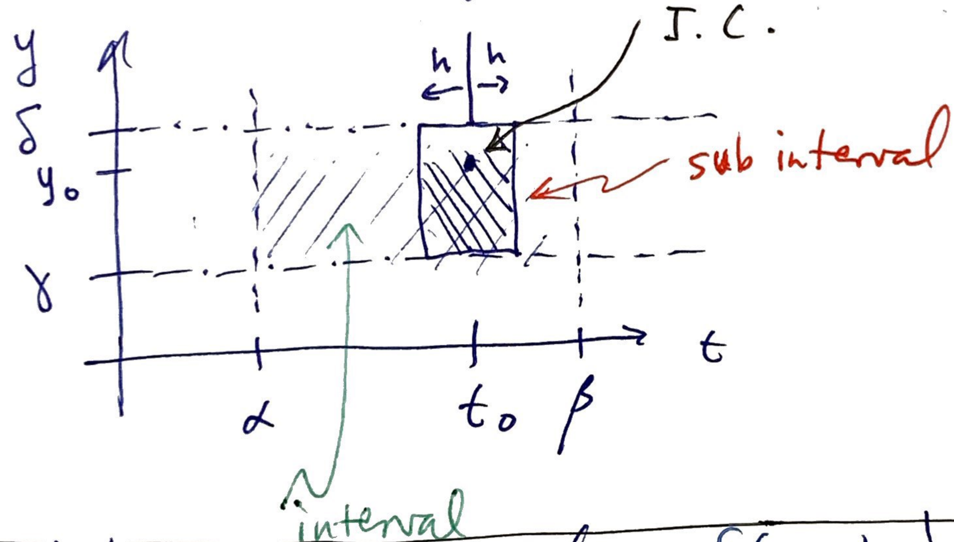
Consider $\boxed{\text{Non-Lin. 1st order ODEs}}$:

**Thm** For the ODE $y' = f(t,y)$ with $y(t_0)=y_0$

If $f(t,y)$ and $\boxed{\dfrac{\partial f}{\partial y}}$ are continuous in the rectangle $t \in (\alpha, \beta)$ by $y \in (\gamma, \delta)$ which must contain $\boxed{y(t_0)=y_0}$

Then there exists a unique solution in some sub-interval $t_0 - h < t < t_0 + h$

The interval is a rectangle




$\boxed{\gamma, \delta, \alpha, \beta, h \text{ are all based on } f(x,y) \;\&\; (t_0, y_0)}$

$\boxed{EX}$ Analyze the IVP for existence

$$(y' = y^{1/3}) * y^{-1/3}, \quad y(0) = 0$$

- $f(x,y) = y^{1/3}$
- $\dfrac{\partial F(x,y)}{\partial y} = \frac{1}{3} y^{-2/3}$

$f$ = cont. everywhere

$\dfrac{\partial f}{\partial y}$ is discontinuous @ $y = 0$

avoid $y = 0$

Can we apply the theorem? NO

B/C the I.C. at $t_0 = 0$ is $y_0 = 0$. So we <u>cannot</u> imply uniqueness (even existence).

BTW: Note that we can solve this ODE via separation of variables: (mult. by $y^{-1/3}$)

$$y^{-1/3} dy = dt$$

integrate $\int y^{-1/3} dy = t + C$

$$\boxed{\frac{3}{2} y^{2/3} = t + C}$$ implicit soln

- Apply the I.C. $\cancel{\frac{3}{2} y_0^{2/3}}^{\,0} = \cancel{t_0}^{\,0} + C \Rightarrow \boxed{C = 0}$

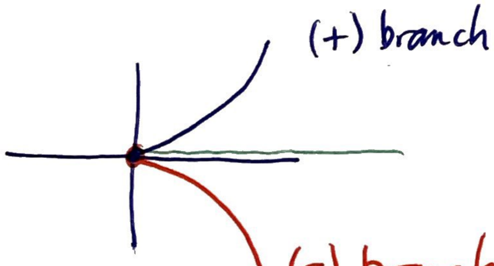
So the solution is $y^{2/3} = \frac{3}{2} t$ ) cube

$y^2 = \left(\frac{3}{2} t\right)^3$ ) square root

two branches

$$\boxed{y(t) = \pm\sqrt{\left(\frac{2}{3} t\right)^3}}$$

not unique




- Furthermore

$(t) \equiv$ is also a s

$\Rightarrow$

**Ex** Determine the intv'l of validity for

$$y' = y^2, \quad y(0) = y_0$$

Note that $y_0$ is not specified.

1st If $y_0 = 0$ then $\boxed{y(t) \equiv 0}$ is a soln to the IVP

2nd Non-Lin Thm's Premise: $f = y^2$ & $\frac{\partial f}{\partial y} = 2y$ are both continuous everywhere

So we have unique solutions no matter $y_0$'s value.

We can solve this IVP to gain further insight

- sep. of vars

$$y^{-2} y' = 1$$

$$\int y^{-2} dy = \int dt$$

$$-\frac{1}{y} = t + C$$

- Apply the I.C.

$$-\frac{1}{y_0} = 0 + C \Rightarrow \boxed{C = -\frac{1}{y_0}}$$

$$\Rightarrow \quad -\frac{1}{y} = t - \frac{1}{y_0}$$

- Solve for $y$ $\Rightarrow$ $\boxed{y(t) = \dfrac{1}{\frac{1}{y_0} - t}}$, $y_0 \neq 0$

$\Rightarrow \boxed{y(t) = \dfrac{y_0}{1 - y_0 t}}$ but $1 - y_0 t = 0$ when $t = 1/y_0$

If $y_0 > 0$ then intv'l of validity is $-\infty < t < \frac{1}{y_0}$

If $y_0 = 0$ then intv'l of validity is $-\infty < t < \infty$

If $y_0 < 0$ then intv'l of validity is $\frac{1}{y_0} < t < \infty$

All outcomes are unique but which outcome to use depends on $y_0$

$\boxed{\text{EX}}$ { 2.2 HW (# 39) } we were asked to solve 

$$y' = (y-1)^2 \quad \text{with } y(0) = 1$$

We separated variables and integrated

$$\boxed{y(x) = -\frac{1}{x+c} + 1}$$

Now lets apply the I C.

$$y(0) = -\frac{1}{0+c} + 1$$

$$1 = -\frac{1}{c} + 1 \qquad \text{we need } -\frac{1}{c} = 0$$

so $c$ is undefined

Since $\bullet\ f(x,y) = (y-1)^2$ is cont. $\forall y$

and $\bullet\ \frac{\partial f}{\partial y} = 2(y-1)$ is cont. $\forall y$

$\Rightarrow$ we are guarenteed a unique soln.

{But that solution depends on $y_0$}

| |
|---|
| We cannot form a solution from $y(x)$ above when $y_0 = 1$ yet there is a solution and it is unique ... So what is it? |

The solution above does not apply if $y_0 = 1$

But we know $\exists$ a unique soln ... by observation

we see that $\boxed{y(x) \equiv 1}$ is our solution

when $y_0 = 1$

Summary: $\begin{cases} x_0 = 0,\ y_0 = 1 : & \text{soln } y(x) \equiv 1 \\ x_0 \neq 0, \text{ no matter } y_0 : & \text{soln } y(x) = -\frac{1}{x+c} + 1 \end{cases}$ {get "c" by applying I.C.

NOTE: Duplicate Terminology...

- In $1^{st}$ order ODE's we also define an ODE to be

homogeneous if it is of the form

$$y' = F(y/x)$$

{as well as if $y' + p(x)\,y = 0$.} ← homog.

$y' + p(x)\,y = g(t)$ ← non-homog.

ex $$y' = \frac{xy - y^2}{x^2}$$

which can be re-writen as

$$y' = \underbrace{\left(\frac{y}{x}\right) - \left(y/x\right)^2}_{= F(y/x)}$$

BTW:

To solve such eqns (i) let $u = y/x$

so then $y = ux$ & so $\boxed{y' = u'x + u} = F(u)$ ← LHS ← RHS

and the ODE becomes a Linear ODE in "u"

(ii) Solve for "u", then (iii) back substitute using $y = u \cdot x$ ← soln.