Ordinary Differential Equations

Chpt 1 1st order ODE's

1.1 Introduction and Classification

- This is an equation: $y = 2x^2 + x - 10$

  function: $f(x) = 2x^2 + x - 10$

- Solving Eqns:

  Ex
  $$y^2 - 4y - 12 = 0$$
  $$(y-6)(y+2) = 0$$
  $$\boxed{y=6} \quad \boxed{y=-2} \text{ solution}$$

- Differential Equations not only contain a variable but that variables rate of change.

  Ex $\quad y' + y = 0 \qquad y' \equiv \dfrac{dy}{dx}$

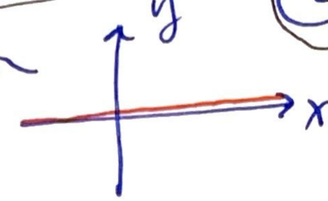
How can we guess a function $y(x)$ such that it's derivative added to the function itself is zero?

Solving such eqns is the topic of this class: **Differential Eqns**

$\boxed{\text{EX}}$ cont. $y'+y=0$

by observation $y \equiv 0$ is a solution

identically zero



$\frac{dy}{dx} = 0$ also and the

D.E. $y' + y = 0$ becomes

$$0 + 0 = 0$$

$y = 0$ is reffered to as the trivia Soln

- The more analytic way is to seperate variables and integrate

$$\frac{dy}{dx} + y = 0$$

$$\rightarrow \frac{dy}{dx} = -y \qquad * dx \text{ \& } \div y$$

A D.E. in differential mode $\rightarrow$ $\boxed{\frac{dy}{y} = -dx}$

integrate

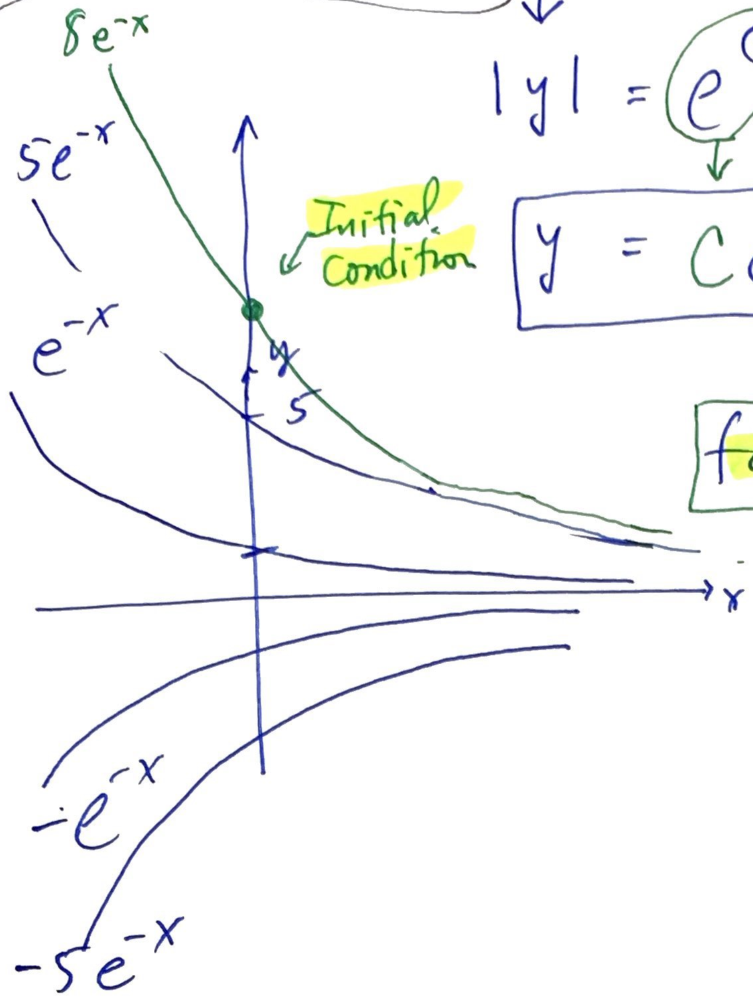
$$\int \frac{dy}{y} = -\int dx$$

soln

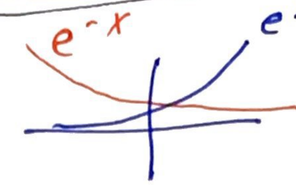
$$\ln|y| = -x + c$$

exponentiate

$$e^{\ln|y|} = e^{-x+c}$$



$$|y| = e^{c} e^{-x}$$



$$\boxed{y = Ce^{-x}}$$ general solution to the ODE $y' + y = 0$

families of solutions

- If we are given a condition such as $y(0) = 8$ then we can determine "C"

$$\longrightarrow \quad y(0) = Ce^{-0}$$

$$8 = C \cdot 1 \implies C = 8$$

$$\boxed{y(x) = 8e^{-x}}$$ specific solution

(*) Solving Differential eqns is just a process of integration

diff'n is a science. Int'n is an Art.

(4)

④ **Nomenclature**

$y = f(x)$

$y' + y = 0$ is an **Ordinary Differential Eqn**

ODE

- if $u = f(x, y, z)$ then if we take the derivative of f w.r.t. only on var, y, say the we call that a parial derivative

(ex) $u = xy^2 + z^2$

$$\frac{\partial u}{\partial y} = \frac{\partial(xy^2 + z^2)}{\partial y}$$

$$= x\frac{\partial y^2}{\partial y} + 0$$

$$\boxed{\frac{\partial u}{\partial y} = 2xy}$$

likewise $\frac{\partial u}{\partial x} = y^2$ & $\frac{\partial u}{\partial z} = 2z$

- Equations with partial derivatives such as $\frac{\partial u(x,y,z)}{\partial y} + u = \frac{\partial u(x,y,z)}{\partial z}$ is called a

partial differential Equation PDE

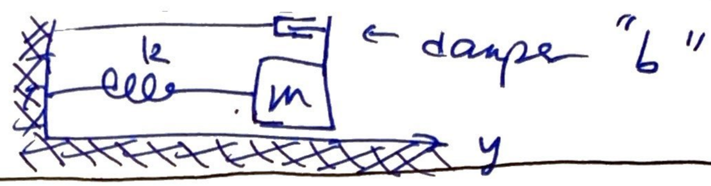
$u = F(x, y, z)$

⊛ **Order:** • An ODE with the highest derivative of y' is 1^{st} order ODE.

ex $y' + y = 0$ 1^{st} order ODE

• y'' is 2^{nd} order

ex $y'' + \frac{b}{m} y' + \frac{k}{m} y = 0$ 2^{nd} order ODE



• $y^{(iv)}$ is 4^{th} order

⊛ **Linear** vs. non-linear

$y' + y = 0$ is linear

$y' + y^2 = 0$ is non-linear

$y' + \sin(y) = 0$ is non-linear

independant var.

$y' + x^2 y = 0$ is linear

dependent variable

Recall $y = f(x)$

dep. var

independent var.

ex

$y'' + \sin(x)\, y = 0$ linear 2^{nd} order ODE

$(y'')^2 + \sin(x)\, y = 0$ 2^{nd} order Non-linear

$y''' + y \cdot y'' + y' + y = 0$ Non lin 3^{rd} order

$y^{(iv)} + \sin(y) = 0$ Non-linear 4^{th} order

⊗ Initial Value Problems (IVP)

- ODE's solutions use constants of integration

EX $y'' + 1 = 0$

$\rightarrow \quad y'' = -1$ use defn.

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = -1$$

$$d\left(\frac{dy}{dx}\right) = -dx$$ integrate

$$\int d\left(\frac{dy}{dx}\right) = -\int dx$$

$$y' = -x + c$$

$$\frac{dy}{dx} = -x + c$$

$$dy = (-x + c)\,dx$$ integ.

$$\boxed{y = -\frac{x^2}{2} + cx + d}$$ gen. solun.

1^{st} order ODE's have gen. solutions with one constant

2^{nd} order ODE's have gen. solns with 2 const

n^{th} order ODE's have n-conditions

- An $\boxed{\text{initial value problem}}$ (IVP) is an ODE with sufficient data to pin down one soln.

EX

IVP.

$$y' + y = 0, \quad y(0) = 8$$

$$y'' + y = 0, \quad y(0) = 8, \; y'(0) = -1$$

$$\underbrace{y''' - y = 0}_{\text{ODE's}}, \; \underbrace{y(1) = 5, \; y'(1) = -2, \; y''(1) = 0}_{\text{"Initial" Conditions}}$$

Both give $\Rightarrow$ Specific Solution

⊛ BVP are $\boxed{\text{Boundary Value Problems}}$

EX $y'' + 3y' - 2y = 0 \quad y(0) = 10$ & $y(3) = -2$

No conditions involving derivatives only different locations, then we call the ODE & Conditions a Boundary Value Problem (BVP)

Implicit vs. Explicit Solutions

- Recall from Calculus an equation such as

$y^2 + \sin(x^2) = 1$ is an implicit eqn.

vs. $y = \pm\sqrt{1 - \sin(x^2)}$ is an explicit eqn

- Sometimes an ODE cannot be solved for the dependent variable.

Ex $\frac{dy}{dx} = -\frac{x}{y}$ Solve using "separation of variables"

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$x^2 + y^2 = 2C$$

$$\boxed{x^2 + y^2 = C^2}$$ ← implicit solution

$$y = \pm\sqrt{C^2 - x^2}$$ ← explicit soln.

families of solutions

$y = ce^{-x}$ ← one parameter family

$y = ax^2 + bx + c$ ← three parameter family

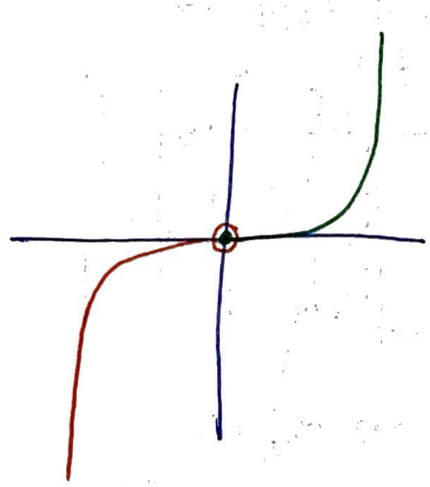
- in general an n-th order ODE has a solution family with n-parameters.

- To fully determine these n-parameters we need n-initial conditions.

piece-wise solutions

ex $xy' - 4y = 0$ this has a solution.

$$y = \begin{cases} -x^4 & x < 0 \\ x^4 & x \geq 0 \end{cases}$$



⊛ prove a solution is valid

(ex) show the implicit relationship

$$y^2+(x^2+1)y-3x^3=C$$

is a solution to the ODE

$$(2xy-9x^2)dx+(2y+x^2+1)dy=0$$

Implicitly differentiate

$$2y\,dy+\underbrace{d(x^2+1)\cdot y+(x^2+1)\cdot dy}_{\text{product rule}}-3\cdot 3x^2dx=0$$

$$2y\,dy+2x\,dx\cdot y+(x^2+1)\cdot dy-9x^2dx=0$$

group

$$(2xy-9x^2)dx+(2y+x^2+1)dy=0$$

same ODE ✓

Differential Equations

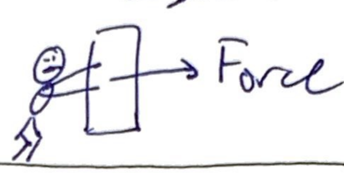
1.1 Introduction

An equation that involves a variable and the rate's of change of that variable is called a differential equation.

Q: where might Diff. Eqns come from?

Ex From Newton's Law in physics

$$F = ma$$



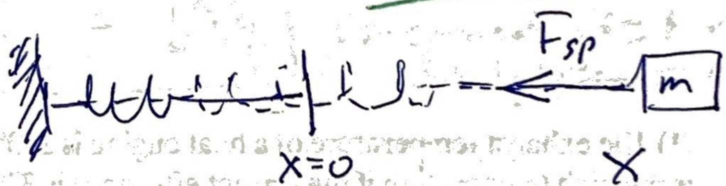
but $a = \frac{d\,\text{velocity}}{d\,\text{time}} \rightarrow a = \dot{v}$

but $v = \frac{d\,\text{position}}{d\,\text{time}} \rightarrow v = \dot{x}$

$\Big\}\ a = \ddot{x} = \frac{d^2x}{dt^2}$

Then the Force eqn becomes $F = m\ddot{x}$

• Now if the force is a spring we know from Hooke's Law that $F_{sp} = -kx$



The eqn becomes $\boxed{\begin{aligned} F_{sp} &= ma \\ -kx &= m\ddot{x} \end{aligned}}$

Reconfigure to get $m\ddot{x} + kx = 0$

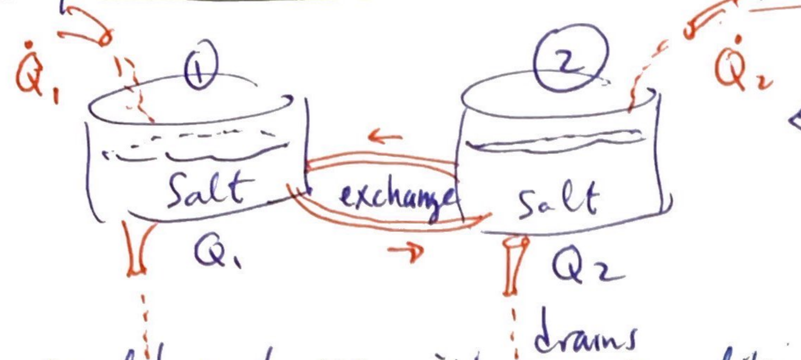
$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Ordinary 2nd order Differential Eqns

• Finally lets introduce a system of Differential Eqns.

Systems of ODE's.

EX



Here:
Q_1 = total salt in tank 1
Q_2 = total salt in tank 2
(once water is evaporated off)

• We could end up with a system like

$$\begin{cases} \dfrac{dQ_1}{dt} = 4Q_1 + 3Q_2 \\[2ex] \dfrac{dQ_2}{dt} = 6Q_2 + Q_1 \end{cases}$$

system of OD Eqns.

let $\vec{q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $\dot{\vec{q}} = \begin{pmatrix} \dfrac{dQ_1}{dt} \\[1ex] \dfrac{dQ_2}{dt} \end{pmatrix}$

• Then the system can be written in vector-matrix form:

$$\frac{d}{dt}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 3 \\ 6 & 1 \end{pmatrix}}_{A}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

OR $\boxed{\dot{\vec{q}} = A\vec{q}}$ condensed vector version of the system of eqns

• Soln might look like

$$\begin{cases} Q_1(t) = a e^{2t} + b e^{-t} \\ Q_2(t) = c e^{2t} + d e^{-t} \end{cases}$$

• Use I.C. $\begin{cases} Q_1(0) = 100 \\ Q_2(0) = 200 \end{cases}$ to determine the a, b, c, d constants.

$\Rightarrow$ complete solution