8.4a Matrices and Determinants (Zill B.2 & B.1)

- matrix $A_{n\times m} = \begin{array}{c} \\ n \text{ rows} \end{array} \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}}^{m \text{ columns}}$

If $n=1$ or $m=1$ we call the matrix a vector.

Ex

$$\vec{V} = \begin{pmatrix} 3t^2 - 2e^t \\ t^2 + 7t \\ 5t \end{pmatrix} = \begin{pmatrix} 3t^2 \\ t^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7t \\ 5t \end{pmatrix} + \begin{pmatrix} -2e^t \\ 0 \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} e^t}$$

Identity matrix $\mathbb{I}$

$$A\mathbb{I} = A \quad \text{or} \quad \mathbb{I}A = A$$

$$\mathbb{I}_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$ diagonal of 1's

Ex

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cdots$$

Determinants (only works on square matrices)

$\det(\mathbb{A}) =$ and $\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(\ ) \equiv |\ |$

then $\boxed{\det(\mathbb{A}) \equiv ad - cb}$

Ex $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1\cdot 4 - 3\cdot 2) = (4-6) = \boxed{-2}$

"Cofactor" expansion

Ex $\det\begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$

row col

$= (-1)^{1+1}\cdot 3 \cdot \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix} + (-1)^{1+2}\cdot 6 \cdot \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix}$

$+ (-1)^{1+3}\cdot 2 \cdot \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix}$

$\det = 3\cdot\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix}$

$= 3\cdot(5\cdot 4 - 2\cdot 1) - 6(2\cdot 4 - (-1)(1)) + 2(2\cdot 2 - (-1)(5))$

$= 3\cdot 18 - 6\cdot 9 + 2\cdot 9$

$= 54 - 54 + 18$

$= 0 + 18$

$= \boxed{18}$

* engineering shortcut (3x3 only)

(iv) subtract.

$$(60 - 6 + 8) - (-10 + 6 + 48) = 62 - 44 = \boxed{18}$$

(ii) multiply up diagonals

$$\left|\begin{array}{ccc|cc} 3 & 6 & 2 & 3 & 6 \\ 2 & 5 & 1 & 2 & 5 \\ -1 & 2 & 4 & -1 & 2 \end{array}\right.$$

(iii) mult. up the anti-diagonals

(i) repeat col 1 & 2

④ matrix inverses

If we have $\mathbb{A}\vec{x} = \vec{y}$

and we want to solve for $\vec{x}$

$2x = y$ solve for y we multiply both sides by the inverse of 2, namely 2^{-1}

$$2^{-1}(2x = y)$$

$$\frac{1}{2} \cdot 2x = \frac{1}{2} y \longrightarrow \boxed{x = y/2}$$

For $\boxed{\mathbb{A}\vec{x} = \vec{y}}$ multiply from the left by the inverse of $\mathbb{A}$: $\mathbb{A}^{-1}$

$$\Longrightarrow \quad \mathbb{A}^{-1}\mathbb{A}\vec{x} = \mathbb{A}^{-1}\vec{y}$$

The inverse has the property $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$

$$\Longrightarrow \quad \mathbb{I}\vec{x} = \mathbb{A}^{-1}\vec{y} \quad \Longrightarrow \quad \boxed{\vec{x} = \mathbb{A}^{-1}\vec{y}}$$

$\mathbb{A}$ must be square

Finding A^{-1}

$\boxed{2\times 2}$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\det(A)}$

(i) exchange the main diagonal
(ii) reverse the signs on the anti-diagonal
(iii) divide by the det (A)

Ex Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ & test it

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}}{2\cdot 5 - 4\cdot 3} = \frac{\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}}{-2} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{4}{2} & -\frac{2}{2} \end{pmatrix}$$

$$\boxed{A^{-1} = \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}}$$

Test: $A^{-1}A$

$$\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$\left(\begin{array}{c|c} -5/2\cdot 2 + 3/2\cdot 4 & -5/2\cdot 3 + \frac{3}{2}\cdot 5 \\ \hline 2\cdot 2 - 1\cdot 4 & 2\cdot 3 - 1\cdot 5 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

⊗ Gaussian Elimination

- Row operations on a matrix (or on eqns)

  (i) we can multiply a row by a constant
  (ii) we can exchange rows in a matrix
  (iii) we can add a multiple of one row to another row and replace the latter row with the results.

- Strategy used in Solving a syst. of eqns via Row Ops:

Ex Solve via Gaussian Elimination the system

$$\begin{aligned} 3x + y + z &= 4 \\ 4x + 2y - z &= 7 \\ x + y - 3z &= 6 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 3 & 1 & 1 & 4 \\ 4 & 2 & -1 & 7 \\ 1 & 1 & -3 & 6 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} a & 0 & 0 & e \\ 0 & b & 0 & f \\ 0 & 0 & c & g \end{array}\right)$$

augmented matrix of the system

$\underbrace{\quad\quad\quad}_{\mathbb{A}}$ $\vec{c}$

→ convert this form back to eqns

$$\begin{aligned} ax + 0y + 0z &= e \\ 0x + by + 0z &= f \\ 0x + 0y + cz &= g \end{aligned}$$

$$\rightarrow \begin{aligned} ax &= e \\ by &= f \\ cz &= g \end{aligned} \longrightarrow \boxed{\begin{aligned} x &= e/a \\ y &= f/b \\ z &= g/c \end{aligned}} \text{ Solution}$$

So the objective is to perform row ops on $\mathbb{A}$ until we have 0's on the off diagonal elements.

- Continuing with the example

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 4 \\ 4 & 2 & -1 & 7 \\ 1 & 1 & -3 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 4 & 2 & -1 & 7 \\ 3 & 1 & 1 & 4 \end{array}\right) *-4 \,;\, *-3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \end{array}\right) *-4 \rightarrow \begin{array}{cccc} (-4 & -4 & 12 & -24) \\ \oplus\; 4 & 2 & -1 & 7 \\ \hline 0 & -2 & 11 & -17 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -2 & 11 & -17 \\ 0 & -2 & 10 & -14 \end{array}\right) *-1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -2 & 11 & -17 \\ 0 & 0 & -1 & 3 \end{array}\right) *11 \,;\, *-3$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & -2 & 0 & 16 \\ 0 & 0 & -1 & 3 \end{array}\right) \div 2$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right) *1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right)$$

$$\begin{aligned} x &= 5 \\ -y &= 8 \\ -z &= 3 \end{aligned}$$

$$\boxed{(x, y, z) = (5, -8, -3)}$$

⊛ Invert matrices with Gaussian Elimination

Strategy:
(i) write $\mathbb{A}$ augmented with $\mathbb{I}$
(ii) perform row ops to turn $\mathbb{A}$ into $\mathbb{I}$
(iii) the RHS is your inverse $\mathbb{A}^{-1}$

i.e. $[\mathbb{A} \mid \mathbb{I}] \xrightarrow[\text{ops}]{\text{row}} [\mathbb{I} \mid \mathbb{A}^{-1}]$

Ex Invert

$$\mathbb{A} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 2 & -1 \\ 1 & 1 & -3 \end{pmatrix}$$

(i)
$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & 1 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} *-4 \,;\, *-3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & 1 \\ 0 & -2 & 11 & 0 & 1 & -4 \\ 0 & -2 & 10 & 1 & 0 & -3 \end{array}\right] \begin{array}{l} \\ *-1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & 1 \\ 0 & -2 & 11 & 0 & 1 & -4 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ *11 \,;\, *-3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -3 & 3 & -2 \\ 0 & -2 & 0 & 11 & -10 & 7 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} *2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & -6 & 6 & -4 \\ 0 & -2 & 0 & 11 & -10 & 7 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} \leftarrow \\ * 1 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5 & -4 & 3 \\ 0 & -2 & 0 & 11 & -10 & 7 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \div -2 \\ \div -1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & -2 & 3/2 \\ 0 & 1 & 0 & -11/2 & 5 & -7/2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

$\underbrace{}_{A^{-1}}$

$$\boxed{A^{-1} = \begin{pmatrix} 5/2 & -2 & 3/2 \\ -11/2 & 5 & -7/2 \\ -1 & 1 & -1 \end{pmatrix}}$$

EX use A^{-1} to solve $\begin{cases} 3x + y + z = 4 \\ 4x + 2y - z = 7 \\ x + y - 3z = 6 \end{cases}$

$$A \cdot \vec{x} = \vec{y}$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 4 & 2 & -1 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$$

So

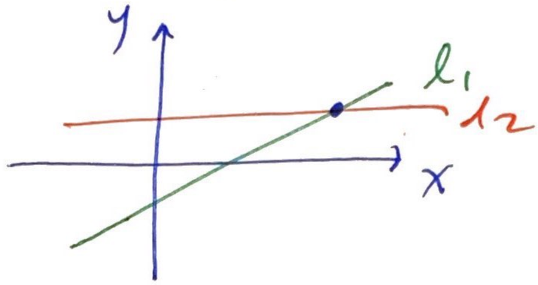
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/2 & -2 & 3/2 \\ -11/2 & 5 & -7/2 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{20}{2} - \frac{28}{2} + \frac{18}{2} \\ -\frac{44}{2} + \frac{70}{2} - \frac{42}{2} \\ -4 + 7 - 6 \end{pmatrix}$$

$$\vec{y} = A^{-1} \cdot \vec{x}$$

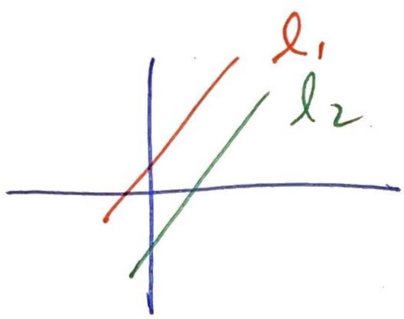
$$= \begin{pmatrix} 10/2 \\ -16/2 \\ -3 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ -8 \\ -3 \end{pmatrix}}$$

(*) Singular Systems

- geometrically solving a system of 2 eqns & 2 unknowns i) Finding the intersection of 2 lines

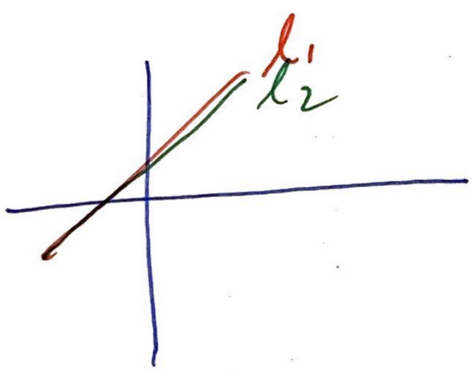


- It is possible that the lines are "||"



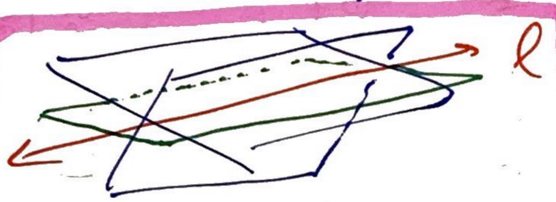
Then no answers exist

- It is possible that the lines are one and the same



Then there are ∞ many answers.

3D The similar situation will exist in higher dimensions

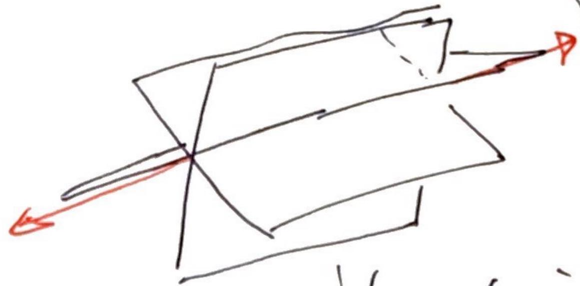


- all three planes are Colinear } → ∞ many solutions in 1 Dim

- all 3 planes are coplanar } ∞ many solution in 2-Dim.

EX

$$5x - 2y + 4z = 10$$
$$x + y + z = 9$$
$$4x - 3y + 3z = 1$$



intersect in a 1-D object: line

$$\rightarrow \left(\begin{array}{ccc|c} 5 & -2 & 4 & 10 \\ 1 & 1 & 1 & 9 \\ 4 & -3 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 0 & 6/7 & 4 \\ 0 & 1 & 1/7 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow ??$$

$$\rightarrow \left\{\begin{array}{l} x + \frac{6}{7}z = 4 \\ y + \frac{1}{7}z = 5 \end{array}\right\}$$ let $z = t$, a parameter

$$\left\{\begin{array}{l} x = 4 - \frac{6}{7}t \\ y = 5 - \frac{1}{7}t \\ z = t \end{array}\right.$$

vector form
$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -6/7 \\ -1/7 \\ 1 \end{pmatrix} t}$$
answer

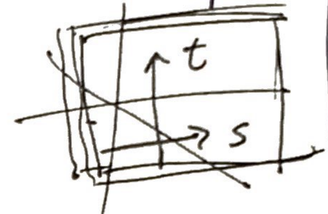
this is a line through the point $(4, 5, 0)$ with a direction of $(6, 1, -7)$

A 1-Dim ∞ so only one parameter

EX 2-D ∞ many solutions

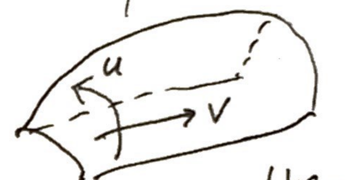
Solve
$$\begin{aligned} x - y - 2z &= 5 \\ 2x - 2y - 4z &= 10 \\ 3x - 3y - 6z &= 15 \end{aligned}$$

All 3 planes are co-planar



$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 5 \\ 2 & -2 & -4 & 10 \\ 3 & -3 & -6 & 15 \end{array}\right) \begin{array}{l} *-2 \,;\, *-3 \\ \\ \\ \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$



non-linear

$$\Rightarrow \quad x - \underset{t}{y} - 2\underset{s}{z} = 5 \quad \Rightarrow \quad \begin{cases} x(s,t) = 5 + t + 2s \\ y = t \\ z = s \end{cases}$$

Solution in parametrized form: 2 params = 2-Dim

$$\boxed{(x(s,t),\ y(s,t),\ z(s,t)) = (2s + t + 5,\ t,\ s)}$$ plane

$s, t \in \mathbb{R} \Rightarrow$ surface, not a curve.