6.4 Derivatives of Logarithmic

I Derivatives of Natural Logarithmic Function

- We know $\ln(x)$ is diff'ble since it is the inverse of the differentiable function $y=e^x$.

Since $\boxed{y=\ln(x) \text{ then } e^y=x}$

- Lets use $e^y = x$ and implicitly differentiate ...

$$\frac{de^y}{dx} = \frac{dx}{dx}$$

$$e^y \frac{dy}{dx} = 1 \longrightarrow \frac{dy}{dx} = \frac{1}{e^y} \quad \text{but } e^y = x$$

so $\frac{dy}{dx} = \frac{1}{x}$ but $y = \ln(x)$

thus we have shown that

$$\boxed{\frac{d\ln(x)}{dx} = \frac{1}{x}} \qquad \frac{de^x}{dx} = e^x$$

- The antiderivative turns out to be ...

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

we will explain the $|x|$ momentarily but first an example

$\boxed{\text{Ex}}$ Diff't $y = \sqrt{\ln(x^2+1)}$

$$y' = \frac{d\sqrt{\ln(x^2+1)}}{dx}$$

$$= \frac{d\left(\ln(x^2+1)\right)^{1/2}}{dx}$$

chain rule

$$= \frac{1}{2}\left[\ln(x^2+1)\right]^{1/2-1} \cdot \frac{d\ln(x^2+1)}{dx}$$

chain rule

$$= \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot \frac{d(x^2+1)}{dx}$$

$$= \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x$$

$$\boxed{y' = \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}}}$$

$\boxed{\text{ex}}$ Find $f'(x)$ if $f(x) = \ln|x|$

Since $\ln|X| = \begin{cases} \ln(X) & x>0 \\ \ln(-x) & x<0 \end{cases}$ $x \neq 0$

$|X| \equiv \begin{cases} X & x \geq 0 \\ -X & x \leq 0 \end{cases}$

So $f'(x) = \begin{cases} \dfrac{d\ln(x)}{dx} & x>0 \\ \dfrac{d\ln(-x)}{dx} & x<0 \end{cases}$

$= \begin{cases} 1/x & x>0 \\ \dfrac{1}{-x}\dfrac{d(-x)}{dx} & x<0 \end{cases}$

$= \begin{cases} \dfrac{1}{x} & x>0 \\ \dfrac{1}{x} & x<0 \end{cases}$

Together $\boxed{\dfrac{d\ln|x|}{dx} = \dfrac{1}{x} \text{ for all } x \neq 0}$ *

This is why the antiderivative leads to there being an absolute value in the answer!

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

* $\boxed{\dfrac{d\ln(x)}{dx} = \dfrac{1}{x} \text{ for } x>0}$

* Recall from 6.2 $\left\{ \frac{d\,a^x}{dx} = f'(0)\,a^x \right.$ if $f = a^x$

Lets revisit this ....

- let $y = a^x$
- take the natural log of both side

$$\ln(y) = \ln(a^x)$$

$$\ln(y) = x\ln(a)$$

a = fixed number

- Now differentiate boths sides w.r.t. x

$$\frac{d\ln(y)}{dx} = \frac{d\,x\ln(a)}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(a)\cancelto{1}{\frac{dx}{dx}}$$

- Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \ln(a)\cdot y$$

- but $y = a^x$

$\ln(e) = 1$

Formula for the derivative of any base exponential

$$\boxed{\frac{d a^x}{dx} = \ln(a)\,a^x}$$

$$\boxed{\frac{de^x}{dx} = e^x}$$

So it turns out that $f'(0)$ is $\ln(a)$

*before we get too far along

Find $\dfrac{d \log_a(x)}{dx}$

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use "change of base"

$$\frac{d\left(\frac{\ln(x)}{\ln(a)}\right)}{dx} = \frac{1}{\ln(a)}\frac{d\ln(x)}{dx} = \frac{1}{\ln(a)}\cdot\frac{1}{x}$$

$$\boxed{\frac{d \log_a(x)}{dx} = \frac{1}{\ln(a)}\cdot\frac{1}{x}}$$

summary of derivatives and integrals...

$$\frac{d e^x}{dx} = e^x \qquad \frac{d\ln(x)}{dx} = \frac{1}{x}$$

$$\frac{d a^x}{dx} = \ln(a)\, a^x \qquad \frac{d \log_a(x)}{dx} = \frac{1}{\ln(a)}\cdot\frac{1}{x}$$

$$\int e^x\,dx = e^x + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \qquad \int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int \log_a(x)\,dx = \frac{1}{\ln a}\left(x\ln x - x\right) + C$$

We study these in math 212

$\boxed{ex}$ Calculate $\int \tan(x)\,dx$

lets write this as $\int \frac{\sin(x)}{\cos(x)}\,dx$

then let
$$\boxed{\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)\,dx \end{aligned}}$$

then the integral becomes

$$-\int \frac{du}{u}$$

$$= -\ln|u| + c$$

unsubstitute to get a new formula to add

$$\boxed{\int \tan(x)\,dx = -\ln|\cos(x)| + c}$$

this can be re-written by using $\ln(a^{-1}) = -\ln(a)$

$$\int \tan(x)\,dx = \ln\left|\frac{1}{\cos(x)}\right| + c$$

$$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + c}$$

official entry.

$\boxed{\text{ex}}$ Consider $y = f(x) = \ln(1-x^2)$

a) Domain $\quad 1-x^2 > 0 \qquad 1 > x^2, \quad x^2 < 1$

$$\boxed{-1 < x < 1}$$

b) x-intercepts:

we seek $f(x)=0$ or $\ln(1-x^2)=0$

this occurs when $1-x^2=1$ so $\boxed{x=0}$

y-intercepts: when $x=0$ we have the y-int. $\quad f(0) = \ln(1-0^2) = 0 \quad \boxed{y=0}$

{ Bottom line: the origin is the only axis intercept.

c) Symmetry?

$$f(-x) = \begin{cases} f(x) & \text{even} \\ -f(x) & \text{odd} \end{cases}$$

$$f(-x) = \ln(1-(-1)^2) = f(x) \quad \boxed{\text{even}}$$

d) Critical points:

$$f'(x) = \frac{1}{1-x^2} \cdot \frac{d\,1-x^2}{dx} = \frac{-2x}{1-x^2}$$

always positive

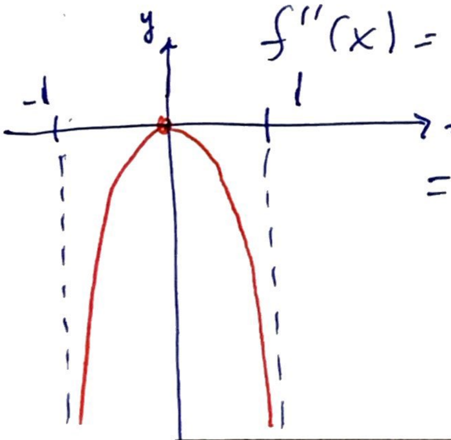
$\boxed{f' = 0 \text{ @ } x=0 \text{ only}}$

signs of f': $x \in (-1,0)$ then $f' > 0$ (increasing)
$x \in (0,1)$ then $f' < 0$ (decreasing)

e) concavities:

$$f''(x) = \frac{(-2x)'(1-x^2) - 2x\cdot(1-x^2)'}{(1-x^2)^2} = \frac{-2(1-x^2)-2x(-2x)}{(1-x^2)^2}$$

$$= -2\left(\frac{1+x^2}{(1-x^2)^2}\right) \text{ always } (-), \quad \boxed{\text{Concave down}}$$



(6.4 to be cont.)

EX Recall $\frac{d\, a^x}{dx} = \ln(a) \cdot a^x$

So $\boxed{\int a^x dx = \frac{a^x}{\ln(a)} + c}$

Now evaluate $\int x e^{x^2} dx$

let $u = x^2$
then $du = 2x\,dx$

$\int x e^{x^2} dx$ $= du/2$

$= \int e^u \frac{du}{2}$

$= \frac{1}{2} \int e^u du$

$= \frac{1}{2} e^u + C$

$= \boxed{\frac{1}{2} e^{x^2} + c}$

$\boxed{\text{II}}$ General Logs & exponentials

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

- Lets look at $\boxed{\dfrac{d \log_a(x)}{dx}}$ (again)

use change of base

$$\frac{d\, \ln(x)/\ln(a)}{dx} = \frac{1}{\ln(a)} \frac{d \ln(x)}{dx} = \boxed{\frac{1}{\ln(a)} \cdot \frac{1}{x}}$$

$\boxed{\text{EX}}$ diff't $y = \log_{10}(\sec x + \tan x)$

$$y' = \frac{1}{\ln(10)} \cdot \frac{d \ln(\sec x + \tan x)}{dx}$$

chain rule

$$= \frac{1}{\ln(10)} \cdot \frac{1}{\sec(x) + \tan(x)} \cdot \frac{d(\sec x + \tan x)}{dx}$$

$$= \frac{1}{\ln(10)} \cdot \frac{1}{\sec(x) + \tan(x)} \cdot \left[\sec(x)\tan(x) + \sec^2(x)\right]$$

$$= \frac{\sec(x)}{\ln(10)} \cdot \frac{1}{\cancel{\sec(x) + \tan(x)}} \cdot \left[\cancel{\tan(x) + \sec(x)}\right]$$

$$\boxed{\frac{d}{dx}\left(\log_{10}(\sec(x) + \tan(x))\right) = \frac{\sec(x)}{\ln(10)}}$$

III Logarithmic Differentiation

To diff't more complicated functions, we may find it easier to take the log then diff't.

> i) Take natural log of both sides
> ii) Diff't implicitly
> iii) Solve for the derivative (y').

Ex diff't $y = \dfrac{e^{-x}\cos^2(x)}{x^2+x+1}$

i) $\ln(y) = \ln(e^{-x}) + \ln(\cos^2(x)) - \ln(x^2+x+1)$

clean up $\ln(y) = -x \cdot \overbrace{\ln(e)}^{1} + 2\ln(\cos(x)) - \ln(x^2+x+1)$

ii) $\dfrac{1}{y}y' = -1 + 2\cdot\dfrac{1}{\cos x}\cdot\dfrac{d\cos(x)}{dx} - \dfrac{1}{x^2+x+1}\cdot(2x+1)$

$\dfrac{y'}{y} = -1 + \overbrace{\dfrac{2}{\cos x}(-\sin x)}^{-\tan x} - \dfrac{2x+1}{x^2+x+1}$

(iii) $y' = y\left\{-1 - 2\tan x - \dfrac{2x+1}{x^2+x+1}\right\}$ ← insert the original "y"

$$\boxed{y' = \underbrace{-\left(\frac{e^{-x}\cos^2(x)}{x^2+x+1}\right)\left\{1 + 2\tan(x) + \frac{2x+1}{x^2+x+1}\right\}}_{f(x)}}$$

(*) we can use logarithmic diff'n to prove the power rule for all n

- Recall: we proved $\boxed{\dfrac{d x^n}{dx} = n x^{n-1} \quad \text{if } n = \text{integer only.}}$

- Now let
$$y = x^n$$
$$\ln(y) = n \ln(x)$$
$$\frac{1}{y} y' = n \frac{1}{x}$$
$$y' = n \cdot \frac{y}{x}$$
$$y' = n \frac{x^n}{x}$$
$$\boxed{y' = n x^{n-1}}$$ $n =$ any real number.

Summary of derivatives of cases with exp & bases:

a) $\dfrac{d a^b}{dx} = 0$, a & b are constants.

b) $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1} f'(x)$, $b = \text{const.}$

c) $\dfrac{d}{dx}(a^{g(x)}) = a^{g(x)} (\ln(a))\, g'(x)$, $a = \text{const.}$

d) $\dfrac{d}{dx}[f(x)]^{g(x)} = f(x)^{g(x)} \left[g(x) \cdot \dfrac{f'(x)}{f(x)} + g'(x) \ln f(x) \right]$

don't memorize - just use log. diff'n

[ex] diff't $y = x^{\tan(x)}$

- $\ln(y) = \tan(x) \cdot \ln(x)$ — product rule
- $\frac{1}{y} y' = \sec^2 x \cdot \ln x + \tan(x) \cdot \frac{1}{x}$
- $y' = y \left[\sec^2(x) \ln(x) + \frac{\tan(x)}{x} \right]$

$$\boxed{y' = x^{\tan(x)} \left[\frac{\tan(x)}{x} + \sec^2(x) \ln(x) \right]}$$

IV The number "e" as a limit

let $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$ & $f'(1) = \frac{1}{1} = 1$

use the definition of derivative

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(x)}{x}$$

insert $f = \ln(x)$

$$f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \cancel{\ln(1)}^{\,0}}{x}$$

$$f'(1) = \lim_{x \to 0} \left(\frac{1}{x}\right) \ln(1+x)$$

$$f'(1) = \lim_{x \to 0} \ln\left[(1+x)^{1/x}\right]$$

$$1 = \lim_{x \to 0} \ln\left[(1+x)^{1/x}\right]$$

$e \qquad e$

$$e^1 = \lim_{x\to 0} e^{\ln(1+x)^{1/x}}$$

$$\boxed{e = \lim_{x\to 0} (1+x)^{1/x}}$$

A new diffinition of the number "e" (pre-calc)

Recall in Calc I we pick "e" to be the base that made diff'n pretty.

Notes:
- the power rule requires that the exponent be a constant
- the exponent rule requires that the base be a constant.

HW Guidelines

* Top Down

all incarnation

please use ( )

must be included

(1)

$$\int (6t^2+5t-4)\,dt$$

"use ="

$$= 6\int t^2\,\underline{\underline{dt}} + 5\int t\,\underline{\underline{dt}} - 4\int \underline{\underline{dt}}$$

$$= \boxed{6\frac{t^3}{3} + 5\frac{t^2}{2} - 4t + C}$$

Box in answers.

* substitutions: If I do not see the actual substitution, you will receive NO credit.

(2)

du

$$\int 10(1-2w)\sqrt{w-w^2}\,dw \quad \begin{cases} u = w - w^2 \\ du = (1-2w)\,\underline{\underline{dw}} \end{cases}$$

don't leave out.

$$= 10\int u\sqrt{u}\,du$$

$$= 10\int u^{3/2}\,du$$

$$= 10\frac{u^{3/2+1}}{3/2+1} + C$$

$$= 10\frac{u^{5/2}}{5/2} + C$$

$$= \frac{10\cdot 2}{5}u^{5/2} + C$$

$$= 4u^{5/2} + C$$

$$= \boxed{4\sqrt{(w-w^2)^5} + C}$$

(* No double columns @ problem level)