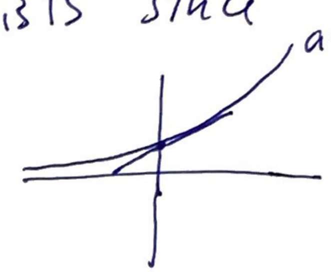
$\boxed{6.3}$ Logarithmic Functions

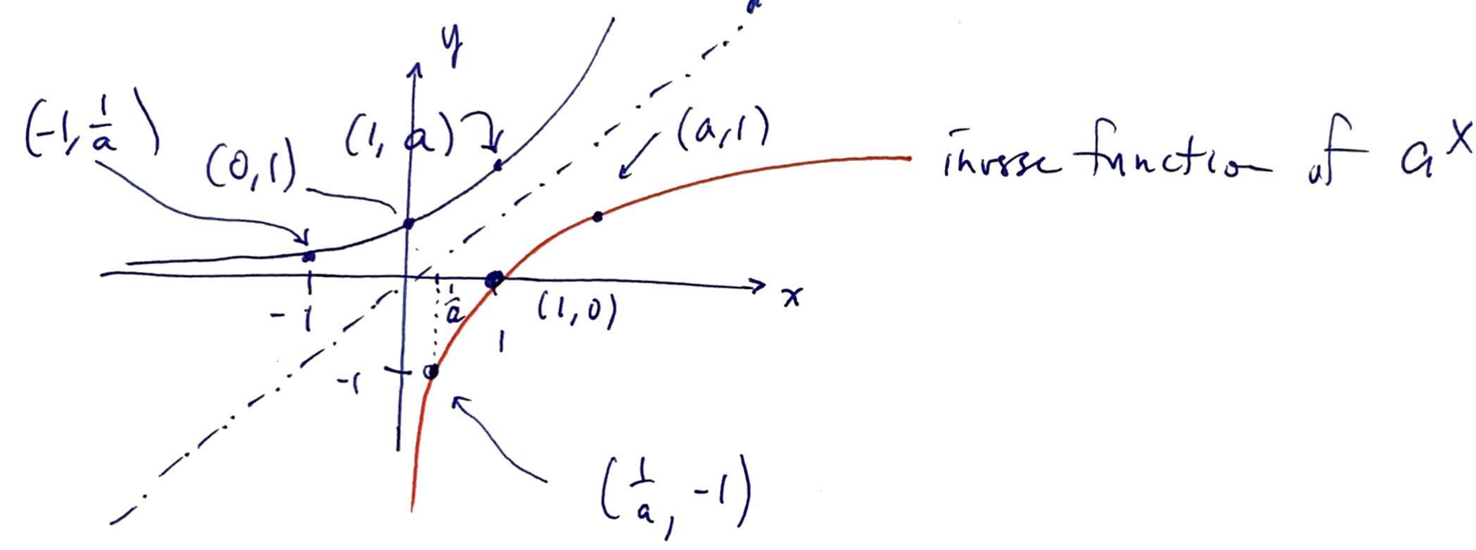
$\boxed{I}$ Log function

The inverse of $f(x)=a^x$ exists since a^x is 1-1 $\{(a^x)' = a^x \cdot \underbrace{f'(0)}_{>0}\}$



a^x is monotonic increasing, thus is 1-to-1.

The inverse function has its graph as being reflected about the line $y=x$



The mathematicians have called this inverse the logarithm function: $\log_a(x) = y \Leftrightarrow a^y = x$

So $\boxed{f(x)=a^x \text{ has an inverse of } f^{-1}(x) = \log_a(x)}$

$= 3^x \rightarrow \log_3(x)$ /// $2^x \rightarrow \log_2(x)$ ...

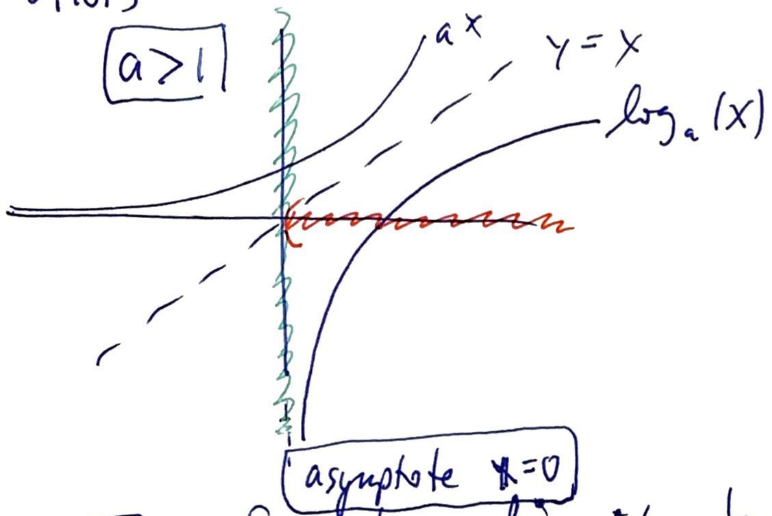
- Domain of $\log_a x$:

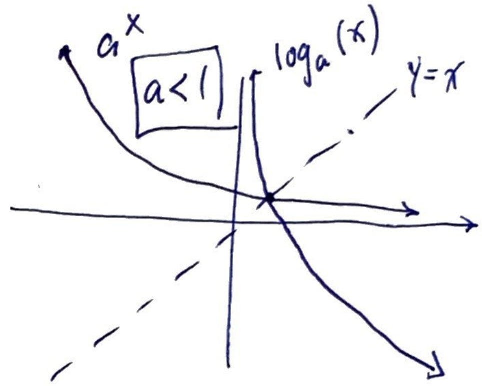
$$\boxed{\begin{array}{l} D_{f^{-1}} = R_f \\ R_{f^{-1}} = D_f \end{array}}$$

so

$$\boxed{D_{\log_a(x)} = \{x \mid x > 0\}}, \quad \boxed{R_{\log_a(x)} = \{y \mid y \in \mathbb{R}\}}$$

- Plots





- Therefore these limits hold

$$\left\{\begin{array}{l} \lim\limits_{x\to 0^+} \log_a(x) = -\infty \\ \lim\limits_{x\to\infty} \log_a(x) = +\infty \end{array}\right. \qquad a > 1$$

$\boxed{\text{Ex}}$ Find $\lim\limits_{x\to 0^+} \log_{10}(\tan^2 x)$

$$= \log_{10}\Big(\lim_{x\to 0^+} \overset{0}{\cancel{\tan^2 x}}\Big)$$

$$= \underline{\underline{-\infty}}$$

(To be cont.)

* properties

since	then
$a^0 = 1$	$\log_a(1) = 0$
$a^1 = a$	$\log_a(a) = 1$
$f(f^{-1}(x)) = x$	$a^{\log_a(x)} = x$
$f^{-1}(f(x)) = x$	$\log_a(a^x) = x$
	$\boxed{\log_a(x^r) = r \log_a(x)}$

proof:

- let $m = \log_a(x) \longrightarrow a^m = x$
- raise both sides to the power of r

$$(a^m)^r = x^r$$

- take $\log_a(\,)$ of both sides

$$\log_a(a^{mr}) = \log_a(x^r)$$

- so $\quad mr = \log_a(x^r)$
- replace $m = \log_a(x)$

$$\log_a(x) \cdot r = \log_a(x^r)$$

Q.E.D.

$$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$$

proof:
- $\begin{cases} \text{let } m = \log_a(x) \longrightarrow a^m = x \\ \text{let } n = \log_a(y) \longrightarrow a^n = y \end{cases}$ $\quad m, n \in \mathbb{R}$
- form $x \cdot y$

$$xy = a^m a^n \Rightarrow xy = a^{m+n}$$

- $\log_a(\ )$

$$\log_a(xy) = \log_a(a^{m+n})$$
$$\log_a(xy) = m + n$$
$$\underline{\log_a(xy) = \log_a(x) + \log_a(y)}. \quad \text{Q.E.D.}$$

$$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)}$$

one way to show this is mimic the above proof
also, however, let $y = Y^{-1}$ and use the product rule

$$\log_a(xY^{-1}) = \log_a(x) + \log_a(Y^{-1})$$
$$= \log_a(x) + (-1)\log_a(Y)$$
$$\log_a\left(\frac{x}{Y}\right) = \log_a(x) - \log_a(Y)$$

so ... corollary

$$\boxed{\log_a\left(\frac{1}{x}\right) = -\log_a(x)}$$

II. Natural Logarithms

We introduced "e" in the exponent section 6.2

$$e^y = x \longleftrightarrow \log_e(x) = y$$

- Here we introduce ln to replace $\log_e$

{ BTW: we let log, without a subscript, to replace $\log_{10}$ }

{ WARNING: some mathematical communities assume that the only logarithm one would ever use is $\log_e$ ... this community then writes log to represent $\log_e$ or ln }

- properties still hold, for example $e^{\ln(x)} = x$

⊛ change of base: Most scientific calculators use only $\boxed{\ln}$ or $\boxed{\log}$ so if you need $\log_3(x)$ you use the change of Base

$$\boxed{\log_a(x) = \frac{\log_b(x)}{\log_b(a)}}$$

specifically $\boxed{\log_a(X) = \frac{\ln(x)}{\ln(a)}}$

or $\boxed{\log_a(x) = \frac{\log(x)}{\log(a)}}$ for Engineers & Scientists.

EX Solve eqn $\ln(x) + \ln(x-1) = 1$

$\underbrace{\ln(x) + \ln(x-1)}= 1$ combine with product rule

$\ln(x(x-1)) = 1$ raise as exponents to base "e"

$e \qquad e$

use a function - functn invs rule

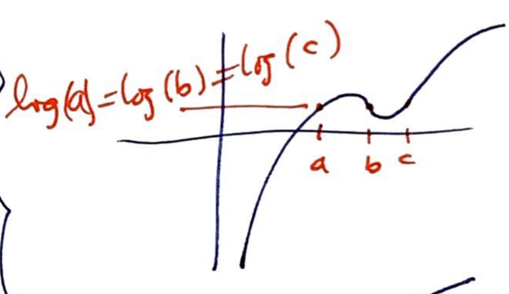
$x(x-1) = e^1$

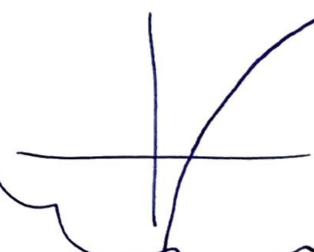
If $e^a = e^b$ then $a = b$

If $\log_a(x) = \log_a(y)$ then $x = y$

b/c both e^x, $\log x$ are monotonic.

If not





OR $x^2 - x - e = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-e)}}{2 \cdot 1}$$

$$x = \frac{1 \pm \sqrt{1+4e}}{2}$$

Q: Do we keep both?

If $x \leq 0$ then $\ln(x)$ is not defined

Q: Is $\sqrt{1+4e} > 1$?

A: yes, clearly.

So we need to exclude the (−) branch.

$$\boxed{x = \frac{1+\sqrt{1+4e}}{2}}$$

Ex evaluate $\log_{\sqrt{e}} e^3$

Base is $\sqrt{e}$, sound like we need to change base

$$\log_{\sqrt{e}}(e^3) = \frac{\ln(e^3)}{\ln(\sqrt{e})}$$

$$= \frac{3}{1/2} = \boxed{6}$$

$$\log_{\sqrt{e}}(e^3) = 6$$

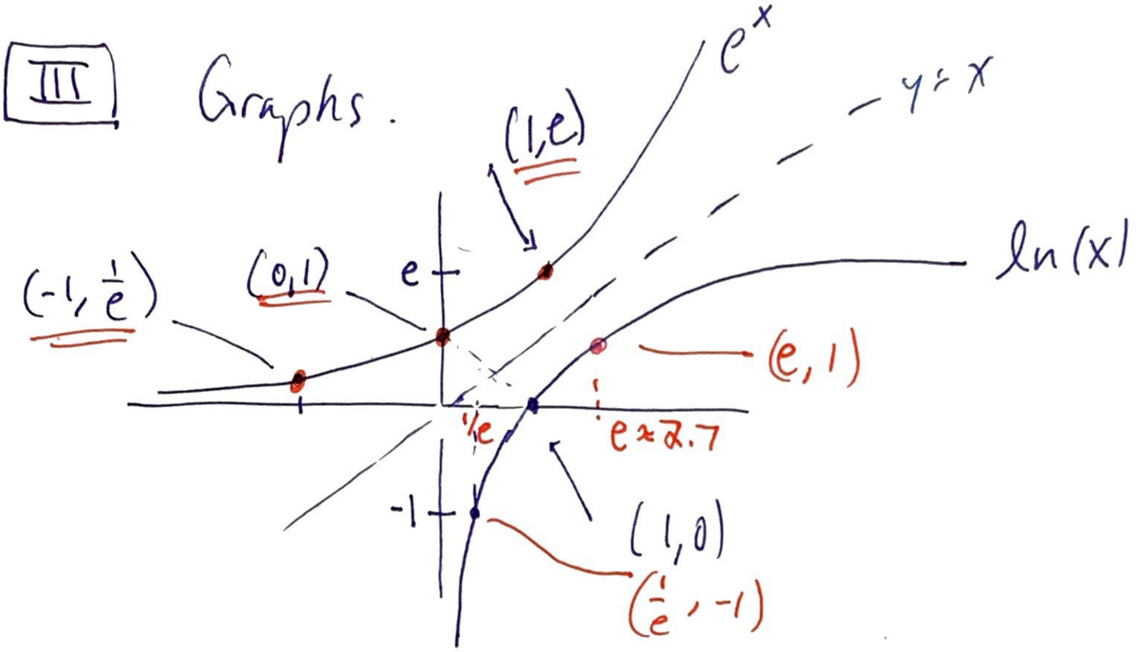
[curious: $\log_a(b) = c \leftrightarrow a^c = b$

$\log_{\sqrt{e}}(e^3) = c \qquad \sqrt{e}^c = e^3$

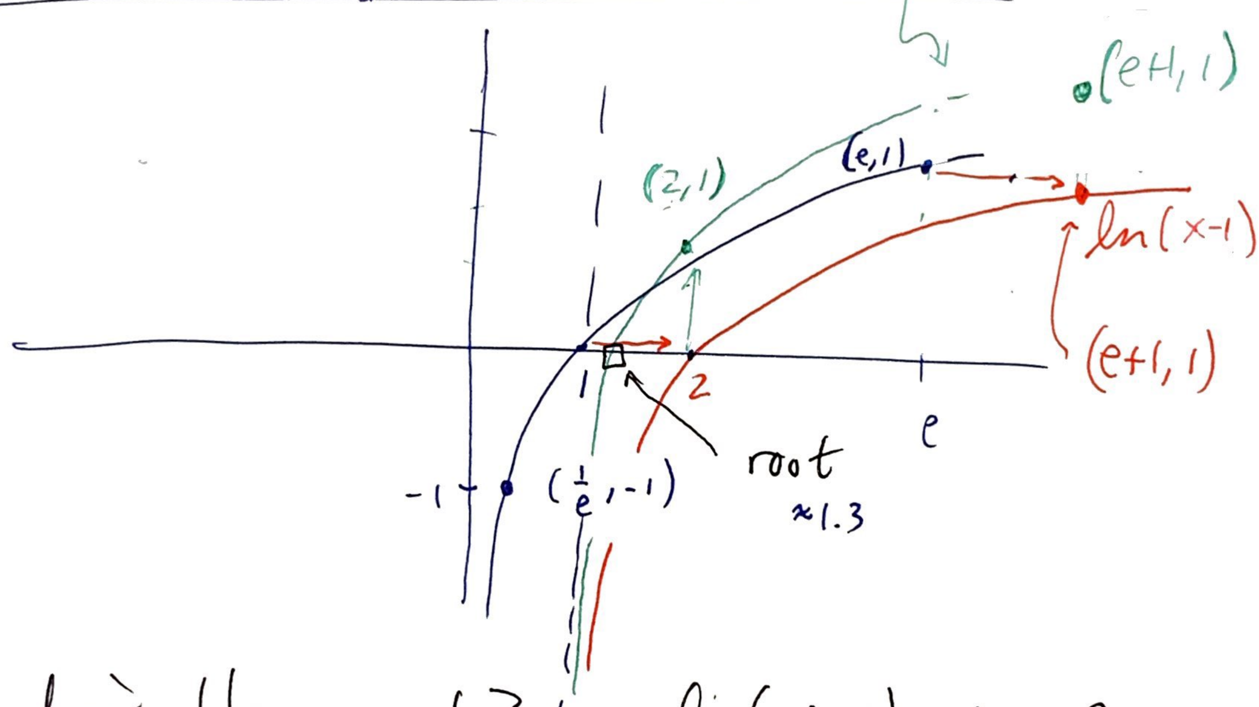
$e^{c/2} = e^3$

$\boxed{c = 6}$]

III Graphs.



Ex sketch $y = \ln(x-1) + 1$



Q: what is the root? $\ln(x-1) + 1 = 0$

$$\ln(x-1) = -1$$

$$e \qquad e$$

$$x - 1 = e^{-1}$$

$$\boxed{x = \frac{1}{e} + 1} \text{ root } \approx 1.3$$

Q: Domain

D: $\{x \mid x > 1\}$

R: $\{y \mid -\infty < y < \infty\}$