6.2 Exponential and their derivatives

I the exponential function

x^2 vs 2^x

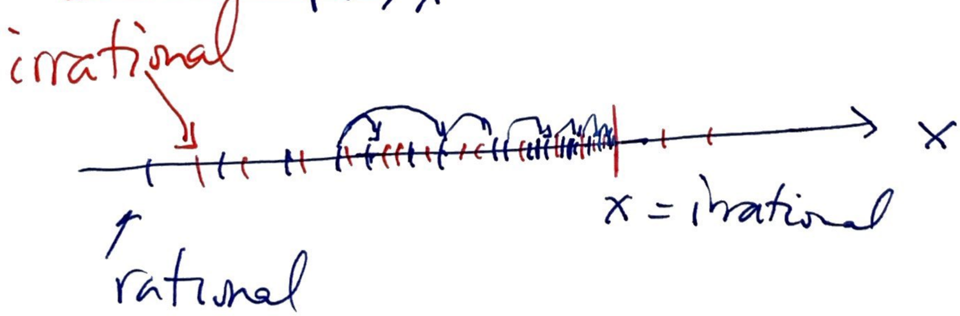
x^2 — polynomial: x is the base

2^x — exponential: x is in the exponent.

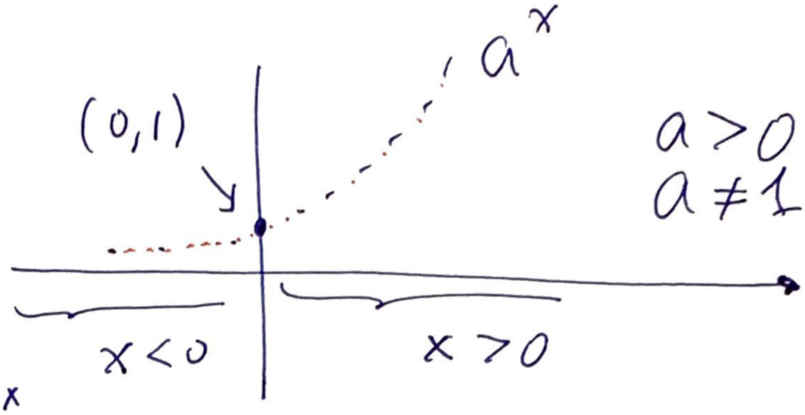
- For $x = n > 0$ then $a^n = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n\text{-times}}$
- For $x = 0$ $\boxed{a^0 \equiv 1}$
- For rational $x = p/q$, p & q are integers, $q \neq 0$

  then $\boxed{a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p}$
- For x an irrational number we define

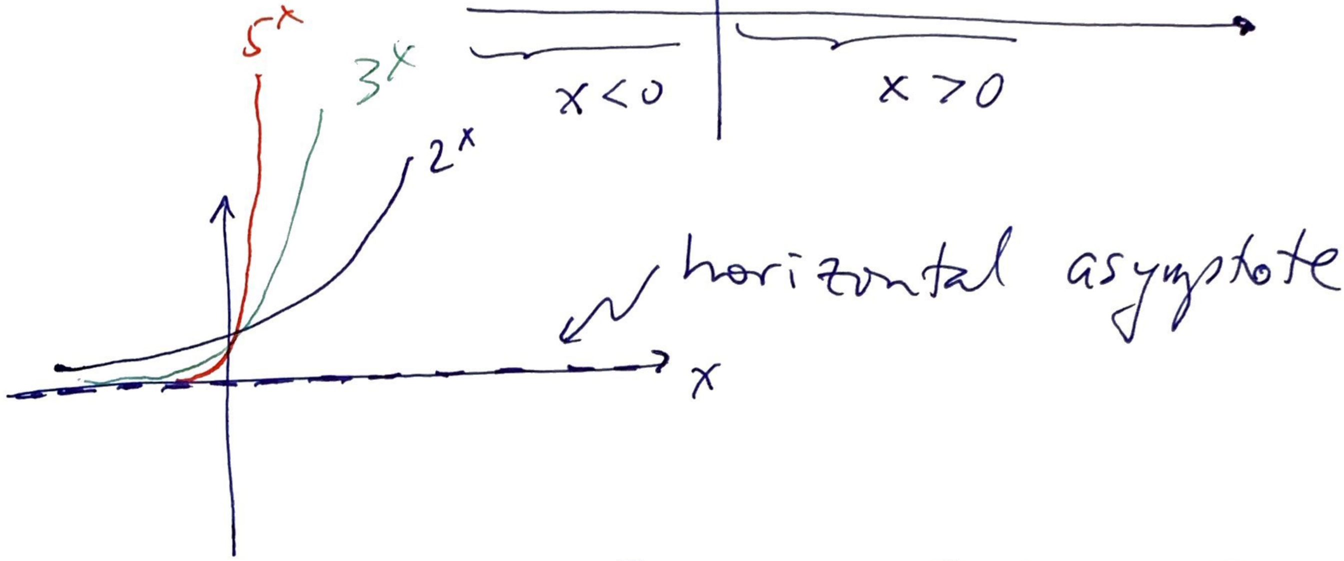
$$\boxed{a^x = \lim_{r \to x} a^r}, \quad r = \text{rational number}$$



* properties



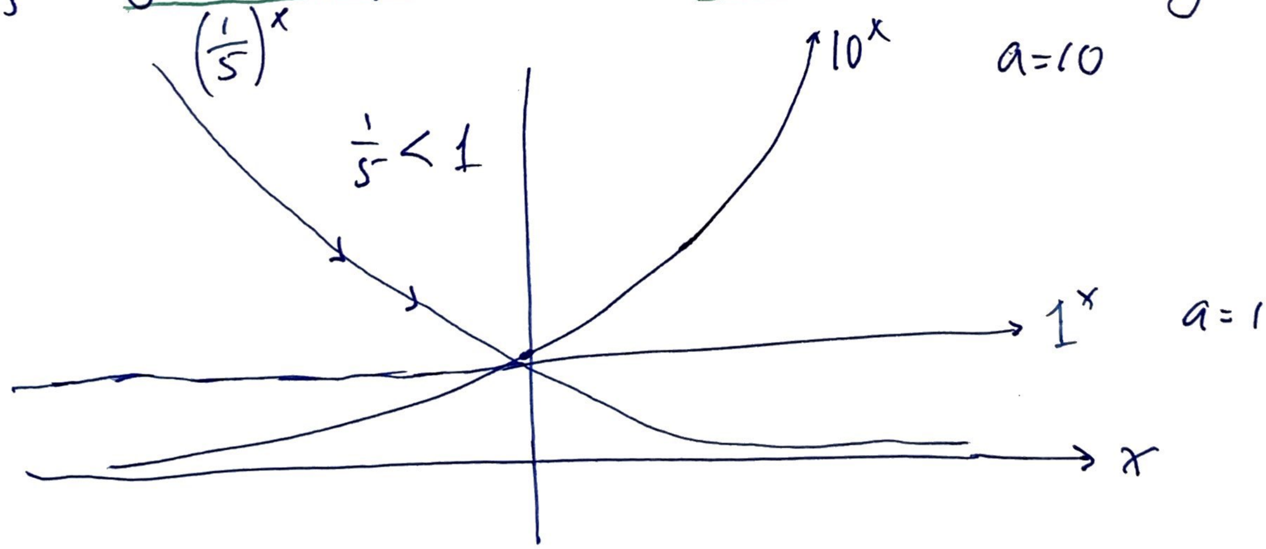
$a>0$
$a\neq 1$



- $D_{a^x}: \{x \mid x \in \mathbb{R}\}$, $R_{a^x}: \{y \mid y>0\}$

  So $\boxed{a^x > 0}$

- If $a>1$ then <u>a^x is increasing</u>.
- If <u>$0<a<1$</u> then <u>a^x is decreasing</u>.



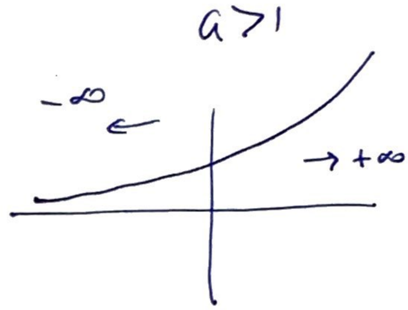
* exponential properties

(i) $a^x a^y = a^{x+y}$

(ii) $a^x / a^y = a^{x-y}$ using $a^{-y} = \frac{1}{a^y}$

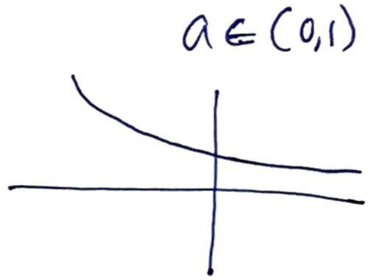
(iii) $(a^x)^y = a^{x \cdot y}$

(iv) $(ab)^x = a^x b^x$



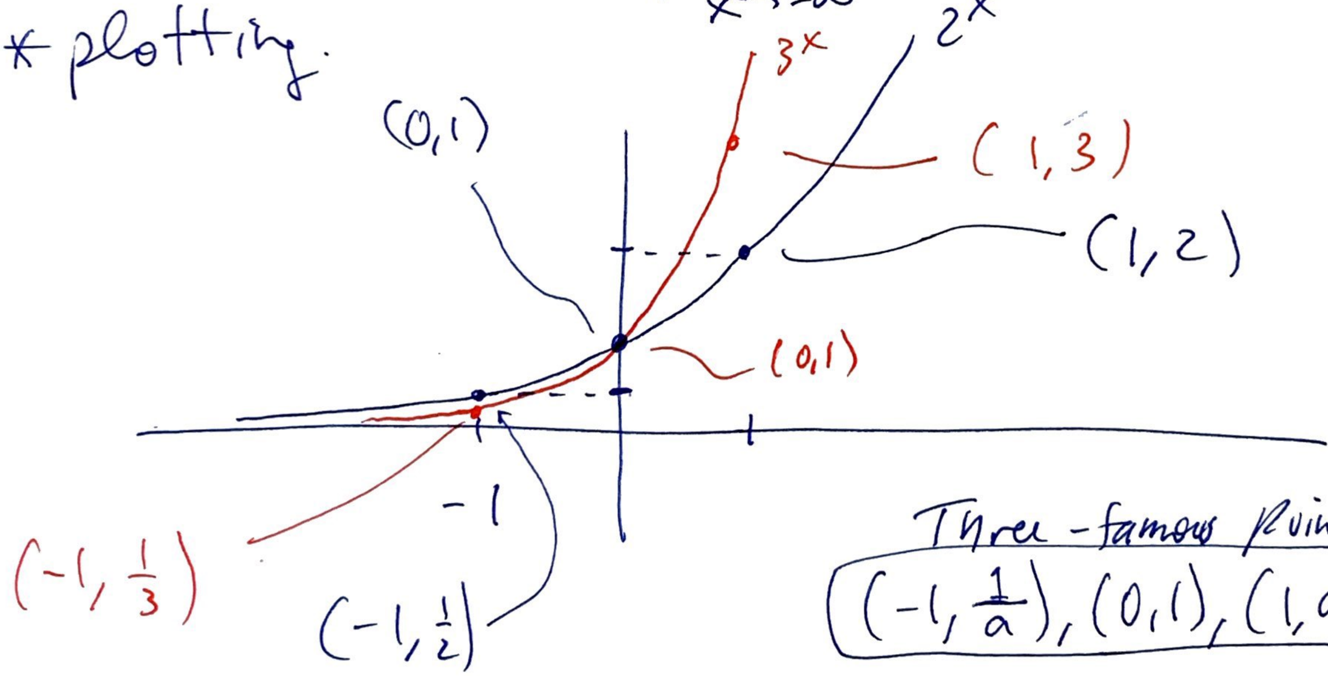
* limits.

for $a>1$: $\begin{cases} \lim\limits_{x \to \infty} a^x = \infty \\ \lim\limits_{x \to -\infty} a^x = 0 \end{cases}$



for $0<a<1$: $\begin{cases} \lim\limits_{x \to \infty} a^x = 0 \\ \lim\limits_{x \to -\infty} a^x = \infty \end{cases}$

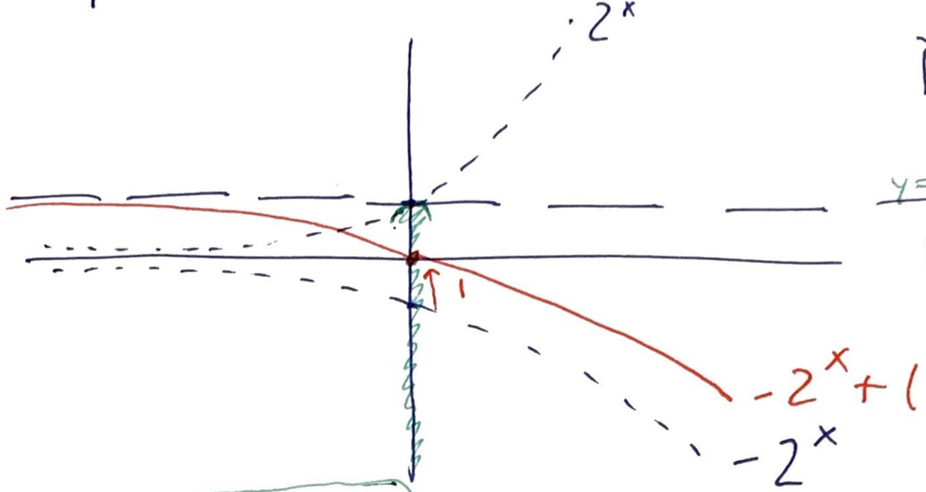
* plotting.



Three-famous points

$\boxed{(-1, \frac{1}{a}), (0,1), (1,a)}$

Ex Graph $f(x) = -2^x + 1$

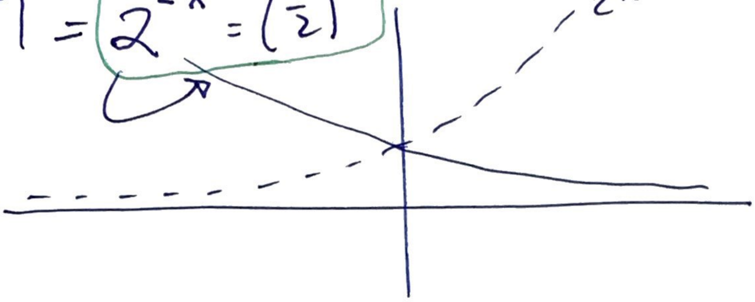


$D_f : \{x \mid x \in \mathbb{R}\}$

$R_f : \{y \mid y < 1\}$

Ex

$g(x) = \boxed{2^{-x} = \left(\frac{1}{2}\right)^x}$



but $2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$, $a < 1$

II Application

Ex A bacterial culture starts w/ 100 bacteria and triples in site every 2 hrs. Fit a model

Table

count	100	300	900	2700
time	$t=0$	$t=2$	$t=4$	$t=6$
model $y = 100 \cdot 3^{t?}$	$100 \cdot 3^0$	$100 \cdot 3^1$	$100 \cdot 3^2$	$100 \cdot 27$ (3^3)
$\boxed{y = 100 \cdot 3^{t/2}}$	$100 \cdot 3^0$	$100 \cdot 3^{2/2}$	$100 \cdot 3^{4/2}$	$100 \cdot 3^{6/2}$

(*) linear growth vs. exponential growth

x	Growth pattern 1		Growth pattern 2	
-6.2	0.62000	$\Delta = 0.031$	0.62000	$\Delta = 0.031$
-2.4	0.65100	$\Delta = 0.031$	0.651000	$\Delta = 0.03255$
1.4	0.68200	$\Delta = 0.031$	0.68355	$\Delta = 0.3418$
5.2	0.71300	$\Delta = 0.31$	0.71773	$\Delta = \ldots$
9.0	0.74400		0.75361	
12.8	0.77500		0.79129	

linear growth

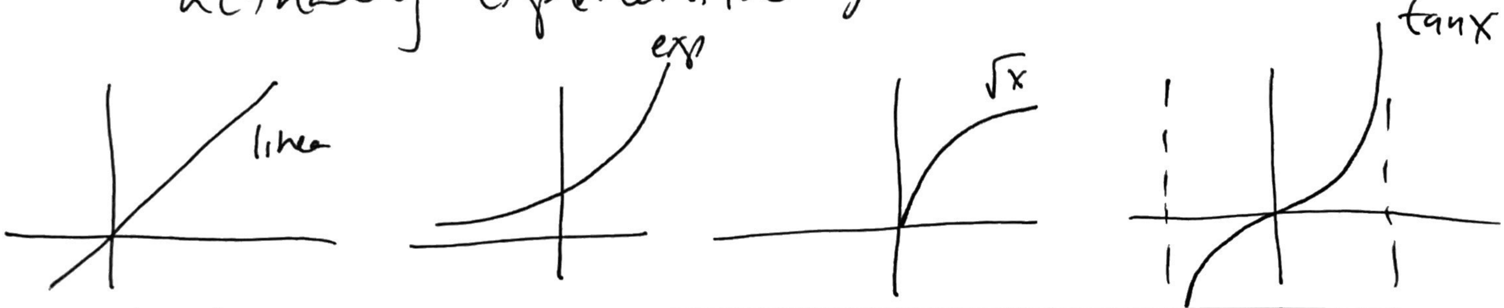
non-linear growth

The non-linear growth might be exponential if the ratio is the same

- $\dfrac{0.62000}{0.651000} = 0.9523809$
- $\dfrac{0.651000}{0.68355}\ 0.9523809$
- $\dfrac{0.68355}{0.71773} = 0.95238$

the ratio is the same (to within acc'y)

This tells us that the non-linear growth is actually exponential growth.



III the derivative of a^x

Using the definition of the derivative: $f(x) = a^x$

$$\frac{da^x}{dx} = \lim_{h\to 0}\left(\frac{a^{x+h} - a^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{a^x a^h - a^x}{h}\right)$$

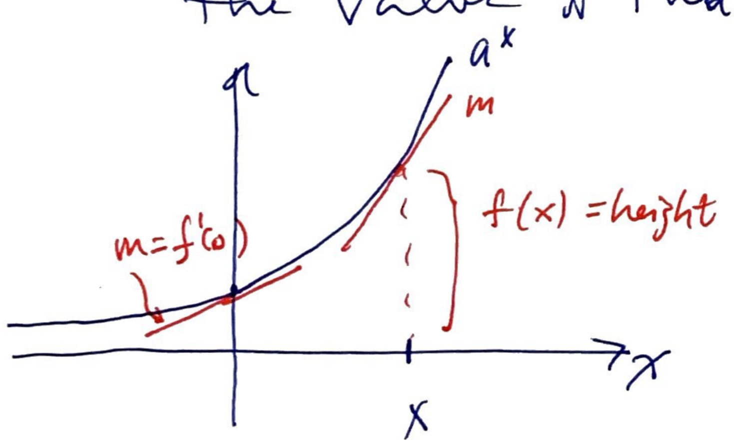
$$= a^x \underbrace{\lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)}_{\left.\frac{da^x}{dx}\right|_{x=0}}$$

$$\left.\frac{df(x)}{dx}\right|_{x=0}$$

$$\boxed{(a^x)' = a^x \cdot f'(0)}$$

$f'(x) = f(x) \cdot f'(0)$ for this particular function $f(x) = a^x$

⊛ This says that the rate of change of growth of an exponential function is proportional to the value of that function, i.e. its height.



Ex bacterial growth: constant of proportion

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = rN$$

$$\int \frac{dN}{N} = \int r\,dt \Rightarrow \ln N = rt + c$$

$$N = e^{rt} e^{c}$$

<u>Bottom Line:</u> to find the value of the derivative of a^x we need to know the value of the derivative of a^x @ $x=0$.

a Catch-22 process

(*) natural number (base) "e"

We define the number "e" as such that

$$\text{the } \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Previously: (in Pre-Calculus) $\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Future: (chpt 11) $\quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$

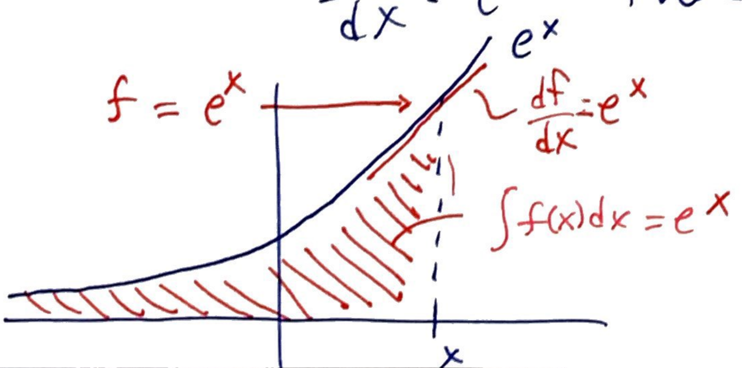
Now $(a^x)' = a^x \cdot f'(0)$ where $f = a^x$

becomes $(e^x)' = e^x \cdot 1$

or just $\boxed{\dfrac{de^x}{dx} = e^x}$ we have a clean formula for at least one base, "e"

BTW: if $\frac{de^x}{dx} = e^x$ then $\int e^x dx = e^x$ also

anti-derivative



Q: What is the value of e?

Pre-Calc: $\left(1+\frac{1}{n}\right)^n$: $n=1$ $\left(1+\frac{1}{1}\right)^1 = 2$

$n=2$ $\left(1+\frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$

$n=3$ $\left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2.37$ etc.

$\vdots$

$n = 10{,}000$ $\left(1+\frac{1}{10{,}000}\right)^{10000} = 2.\underbrace{718}_{\text{3 decimal acc'y}}14$

If we have a calculator that can handle 2^x then we can use a different approach:

- let $(e = 2^c)^x$, then $e^x = 2^{cx}$
- let $f(x) = 2^x$, then $f' = k \cdot 2^x$ where $k = f'(0)$

we can calculate "k" by looking at a table:

h	$(2^h-1)/h$
0.1	0.7177
0.01	0.6956
0.001	0.6934
0.0001	0.6932

So $\lim_{h\to 0}\left(\frac{2^h-1}{h}\right) \approx 0.6932$

$f'(0)$ for $f(x) = 2^x$, that is $f'(0) = k$

we have k to 4 places from the table.

- Now $\frac{de^x}{dx} = \frac{d\,2^{cx}}{dx} = k\,2^{cx}\left(\frac{d\,cx}{dx}\right) = \underline{\underline{ck\,2^{cx}}}$

chain rule ↑

but $\frac{de^x}{dx} = \underline{\underline{e^x}}$

- So $\boxed{e^x = ck2^{cx}}$
  - @ $x=0$ $1 = ck$ so $c = 1/k$
  - @ $x=1$ $\boxed{e = 2^{1/k}} \approx 2^{\frac{1}{0.6932}} \approx \underline{\underline{2.71828}}$

NOTE: $e \approx 2.718281828459045$ to 15 places

EX Find f' & f'' if $f(x) = 2e^x + 5x^2$

$$f' = \frac{d\,2e^x}{dx} + \frac{d\,5x^2}{dx}$$

$$= 2\frac{de^x}{dx} + 5\frac{dx^2}{dx}$$

$$= 2e^x + 5\cdot 2x$$

$$\boxed{f' = 2e^x + 10x}$$

$$f'' = \frac{d(2e^x + 10x)}{dx} = \boxed{2e^x + 10}$$

EX Find y' if $y = e^{-4x}\sin(5x)$

product rule

$$y' = \left(e^{-4x}\right)'\sin(5x) + \left(e^{-4x}\right)\left(\sin(5x)\right)'$$

$$y' = e^{-4x}\left(-\frac{d\,4x}{dx}\right)\sin 5x + e^{-4x}\cos(5x)\left(\frac{d\,5x}{dx}\right)$$

$$y' = e^{-4x}\cdot(-4)\cdot\sin 5x + e^{-4x}\cdot 5\cdot\cos(5x)$$

$$\boxed{y' = e^{-4x}\left(5\cos(5x) - 4\sin(5x)\right)}$$

IV graphing curves with exponentials

EX Sketch $y = -e^{-\frac{1}{x}} + 1$

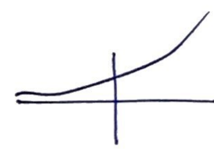
Domain restriction is $\boxed{x \neq 0}$, all other x is OK.

- $\lim_{x\to\infty} \left(-e^{-\frac{1}{x}}+1\right) = -e^{-\lim_{x\to\infty}\left(\frac{1}{x}\right)} + 1$

$= -e^{-0}+1 = -1+1 = \underline{\underline{0}}$ x-axis is a H. Asym.

- $\lim_{x\to-\infty} \left(-e^{-\frac{1}{x}}+1\right) = -1+1 = \underline{\underline{0}}$

- $\lim_{x\to 0} \left(-e^{-\frac{1}{x}}+1\right) = -e^{-\lim_{x\to 0}\left(\frac{1}{x}\right)} + 1$



Better break this up into 0^+ & 0^-

- - $\lim_{x\to 0^+} \left(-e^{-1/x}+1\right) = -e^{-\lim_{x\to 0^+}\left(\frac{1}{x}\right)} + 1 = -\cancel{e^{\text{"}-\infty\text{"}}}^{\,0} + 1 = \boxed{1}$

- - $\lim_{x\to 0^-} \left(-e^{-\frac{1}{x}}+1\right) = -e^{-\lim_{x\to 0^-}\left(\frac{1}{x}\right)} + 1 = -e^{+\infty}+1 = \underline{\underline{-\infty}}$

Q: does $y = -e^{-1/x}+1$ cross the x axis? No (we can't get e^0)

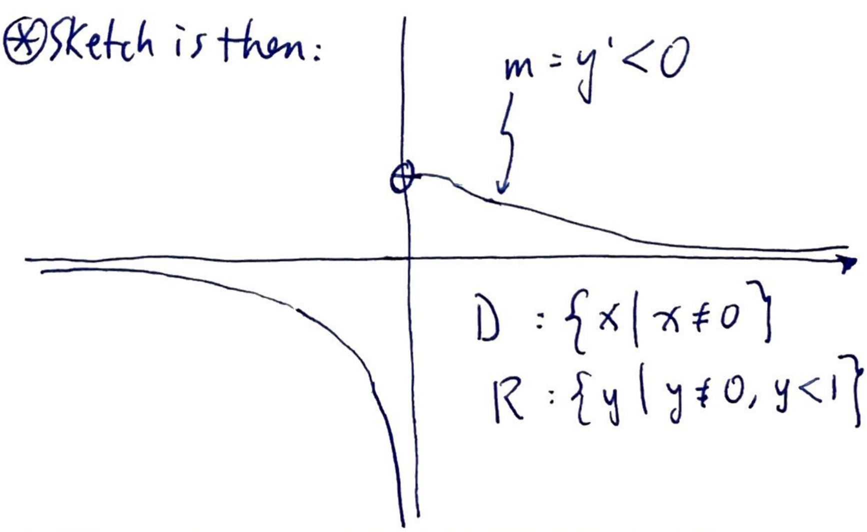
✱ $y' = -\frac{de^{-1/x}}{dx} + 0$

$= -e^{-1/x}\frac{d\left(-\frac{1}{x}\right)}{dx}$

$= e^{-1/x}(-1x^{-2})$

$\boxed{y' = \frac{-e^{-1/x}}{x^2} < 0}$

✱Sketch is then:



D : {x | x ≠ 0}

R : {y | y ≠ 0, y < 1}

V Integration of e^x

As we mentioned, if $\frac{de^x}{dx} = e^x$ then the antiderivative of e^x is e^x.

$$\boxed{\int e^x dx = e^x + c}$$

ex Area under $f(x) = e^x$ from $-\infty$ to x

$$A = \int_{-\infty}^{x} f(t)\,dt$$ ← integration variable

$-\infty$ ← chapter 7 (math 212)

$$= \int_{-\infty}^{x} e^t dt = e^t \Big|_{-\infty}^{x} = e^x - e^{\text{"}-\infty\text{"}} = e^x - 0 = \underline{\underline{e^x}}$$

Ex Find the area under $y = \frac{1}{x^2} e^{-1/x} + 1$ between 1 & 2

$y > 0$ between $x \in [1, 2]$

$$A = \int_{x=1}^{x=2} \left(\frac{1}{x^2} e^{-1/x} + 1\right) dx = \int \frac{1}{x^2} e^{-\frac{1}{x}} dx + \int_1^2 1\,dx = \int_{u=-1}^{u=-1/2} \left(\frac{1}{\cancel{x^2}} e^u\right) \cancel{x^2}\,du + x\Big|_1^2$$

Usubstitution:

let $u = -\frac{1}{x}$

then $du = -(x^{-1})' dx$

$= -(-1x^{-2})\,dx$

$$\boxed{du = \frac{1}{x^2} dx}$$

Transform limits

$x = 1$:

$u = -\frac{1}{1} = -1$

$x = 2$:

$u = -\frac{1}{2}$

$$= \int_{u=-1}^{u=-\frac{1}{2}} e^u du + (2-1)$$

$$= e^u \Big|_{u=-1}^{u=-1/2} + 1$$

$$= e^{-\frac{1}{2}} - e^{-1} + 1$$

$$A = \frac{1}{\sqrt{e}} - \frac{1}{e} + 1$$