Chapter 6 Inverse Functions, Exponential, the Logarithm

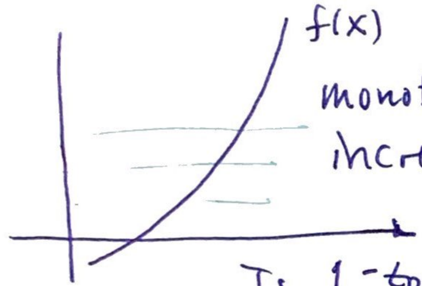
6.1 Inverse Functions (again)

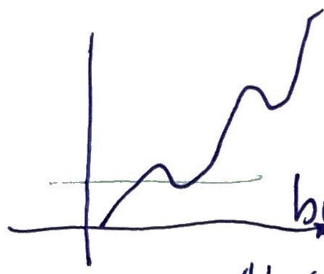
I. Review

A function $f(x)$ is called a one-to-one function if it never takes on the same value twice.

i.e. $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$

- monotonicity, the function needs to be always increasing or decreasing or is a constant.





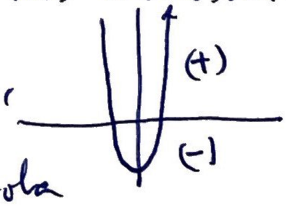
The horizontal line test can be used to determine if a graph is 1-to-1.

Ex Is the function $f(x) = x^3 - x$. Is it 1-1?

Consider the derivative. $f' = 3x^2 - 1$, does this change slopes? If so f is not monotonic, thus NOT 1-1.

Approach

Here f' is a parabola



f is NOT 1-1

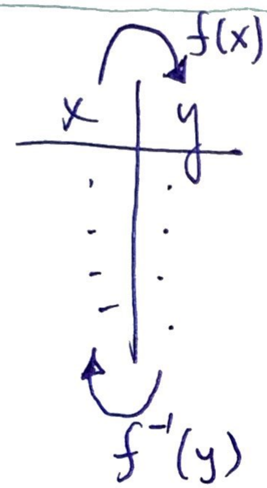
not monotonically increasing / decreasing $\Rightarrow$ Not 1-1.

If $f(x)$ is a 1-1 function with domain A and Range B then its inverse function, $f^{-1}(x)$, is defined by

$$\boxed{f^{-1}(y) = x \text{ iff } f(x) = y}$$

The domain of f^{-1} is B, the range of "f"
The range of f^{-1} is A, the domain of "f"

- Tabularlly



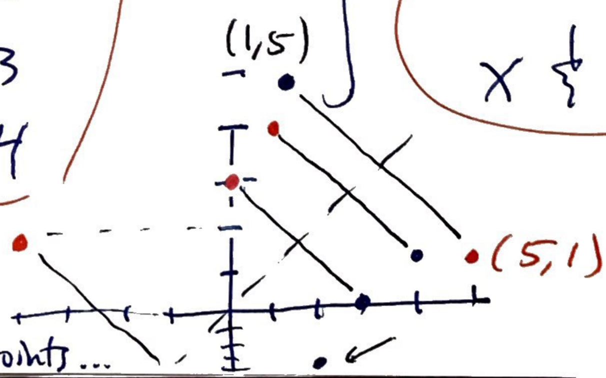
we like to graph f^{-1} in terms of "x" so we use $f^{-1}(x)$

- Steps to find inverses functions
  - Tabularlly

x	$f(x)$
1	5
2	-4
3	0
4	1

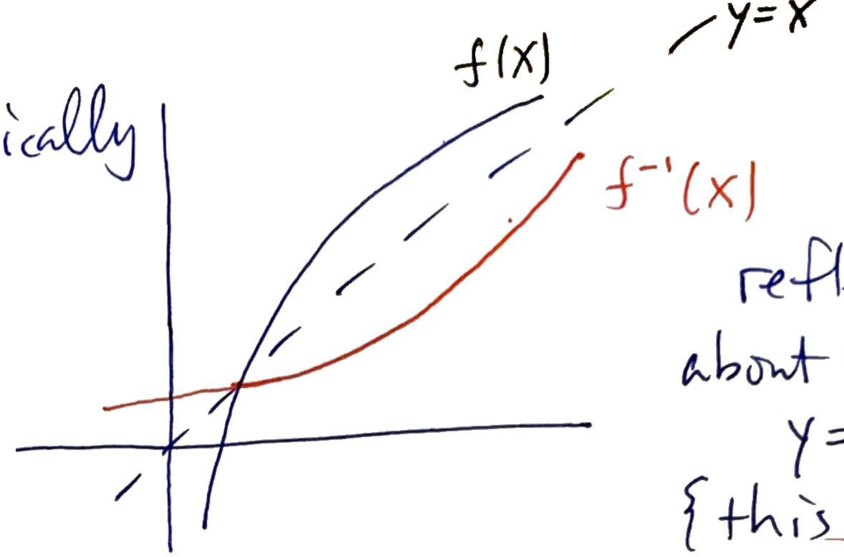
x	$f^{-1}(x)$
5	1
-4	2
0	3
1	4

we exchange x & y



we can see this on a plot of their points...

- graphically



reflect f(x) about the line $y = x$

{this process swap x and y}

- analytically
  1. given $y = f(x)$ swap x & y values
  2. Solve for y
  3. replace y with $f^{-1}(x)$

$\boxed{\text{Ex}}$ Find the inverse function of

$f(x) = 1 + \sqrt{1+x}$, $x \geq -1$

$y = 1 + \sqrt{1+x}$

$x = 1 + \sqrt{1+y}$

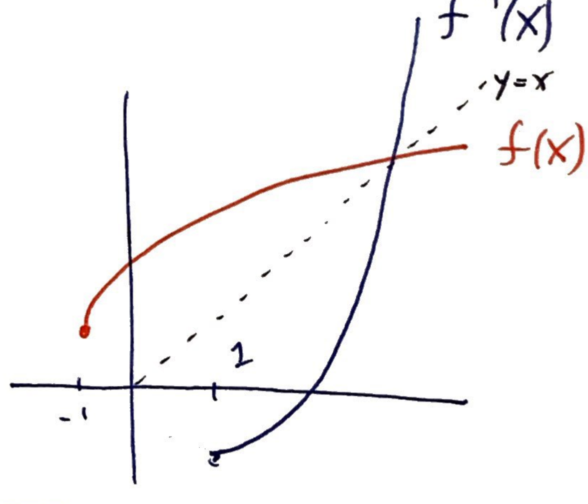
$x - 1 = \sqrt{1+y}$

$(x-1)^2 = 1 + y$

$(x-1)^2 - 1 = y$

or $y = (x-1)^2 - 1$

$\boxed{f^{-1}(x) = (x-1)^2 - 1}$ $x \geq 1$



lets graph f(x) and $f^{-1}(x)$ on desmos...

II The calculus of inverse functions

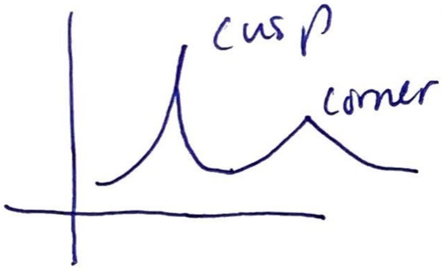
let $f(x)$ be both 1-to-1 and continuous (no break in it)
since the graph of $f^{-1}(x)$ is obtained from the graph of $f(x)$ being reflected across the line $y = x$,
the graph of $f^{-1}(x)$ should have no break either and so must be continuous.

So, intuitively, if $f(x)$ is cont. & 1-1 on I then $f^{-1}(x)$ is cont. & 1-1 over the corresponding interval

Next...

- let $f(x)$ be 1-to-1 and diff'ble

then $f(x)$ has no corners nor cusps in its graph.



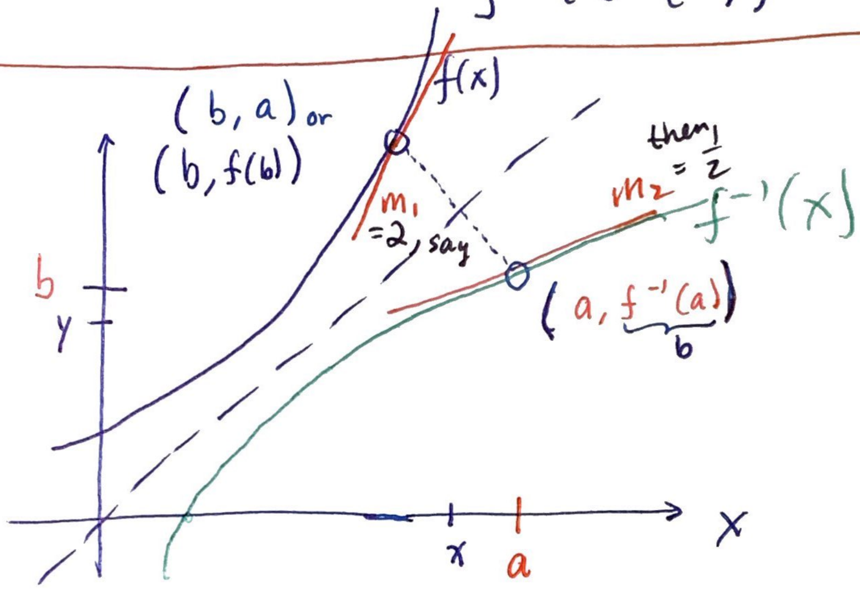
derivatives do not exist @ these cusps and corners.

Then it reasons that a diff'ble function will reflect about the line $y = x$ and produce a curve that is also diff'ble! Thus

$$\boxed{f^{-1} \text{ is diff'ble iff "}f\text{" is diff'ble}}$$

Thm: If $f(x)$ is a one-to-one differentiable function with the inverse function $f^{-1}(x)$

then $$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$ so long as $f'(f^{-1}(a)) \neq 0$



$$\left\{ \begin{array}{l} m_1 = \frac{1}{m_2} \\ m_2 = \frac{1}{m_1} \\ m_1 \cdot m_2 = 1 \end{array} \right.$$

WARNING:

$m_\perp = -\frac{1}{m}$

we do not have the (−) here

proof:

- By def $(f^{-1})'(a) \equiv \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$
- If $f(b) = a$ then $f^{-1}(a) = b$
- If $y = f^{-1}(x)$ then $f(y) = x$ since $f^{-1}(f(x)) = x$
- because f is diff'ble & it is cont. so f^{-1} is continuous & diff'ble too.

So if $x \to a$ then $f^{-1}(x) \to f^{-1}(a)$, that is $y \to b$

- therefore $(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)}$

$\frac{a}{b} = \frac{1}{\frac{b}{a}}$

$$= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} = \frac{1}{\lim_{y \to b} \frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$$

Leibniz notation, the result above

$$\boxed{\left.\frac{dy}{dx}\right|_{x=a} = \frac{1}{\left.\frac{dx}{dy}\right|_{y=f^{-1}(a)}}}$$

- Implict diff'n : if $x = f^{-1}(y)$

$$\frac{dx}{dx} = \frac{d f^{-1}(y)}{dx}$$

$$1 = \frac{d f^{-1}(y)}{dy} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\frac{d f^{-1}(y)}{dy}}$$

$$\boxed{\frac{df}{dx} = \frac{1}{\frac{d f^{-1}(y)}{dy}}}$$

or $\frac{df}{dx} \cdot \frac{d f^{-1}(y)}{dy} = 1$

$$f' \cdot (f^{-1})' = 1$$

Just like in $\perp$ lines

$$m_\perp \cdot m = -1$$

EX

let $f(x) = x^3 + 3\sin(x) + 2\cos(x)$

Find $(f^{-1})'(2)$:

First we need $f'(x) = 3x^2 + 3\cos(x) - 2\sin(x)$

$\{$ note that $f(0) = 2$ then $f^{-1}(2) = 0 \}$

this is the tough part of these problems

so

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

$$= \frac{1}{f'(0)}$$

$$= \frac{1}{3\cdot 0^2 + 3\cos(0) - 2\sin(0)}$$

$$= \boxed{\frac{1}{3}}$$