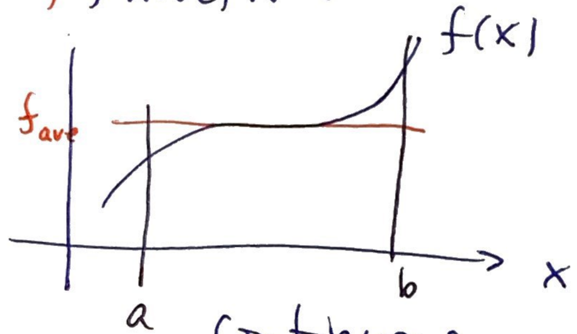
$\boxed{5.5}$ The average value of a function

$\boxed{I}$ f_{ave}

- classically $\quad \bar{y} = \dfrac{y_1 + y_2 + \cdots + y_n}{n}$ (discreet) ← "eggs"

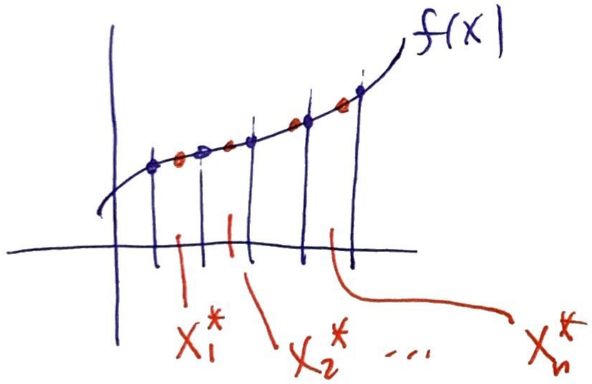
Q: Can we form an average value for a continuous (milk) function

discreet



continuous

Ans: Break the function up into small pieces... and select a sample point within each piece and find f(x) ~~of those~~ sample points...



$\dfrac{a}{b} = \dfrac{1}{\frac{b}{a}}$

- Now

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$ changing continuous into discreet

but $\Delta x = \dfrac{b-a}{n}$ then $n = \dfrac{b-a}{\Delta x}$

so $$f_{ave} \approx \frac{1}{[b-a]} \left\{ f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right\} \Delta x$$

Now as we take smaller and smaller sampling spacings, we have then

$$f_{ave} = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*) \right) \Delta x$$

$$\boxed{f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

This is the average value of a continuous function over [a, b]

$\boxed{\text{EX}}$ Find the average value of $f(x) = (3-2x)^{-2}$ over $[-1, 1]$

Formula $\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$

let $u = 3-2x$
$du = -2\,dx$

$$= \frac{1}{1-(-1)} \int_{x=-1}^{x=1} (3-2x)^{-2}\,dx$$

$u = 3-2(-1)$
$= 3+2 = 5$

$u = 3-2(1)$
$= 3-2 = 1$

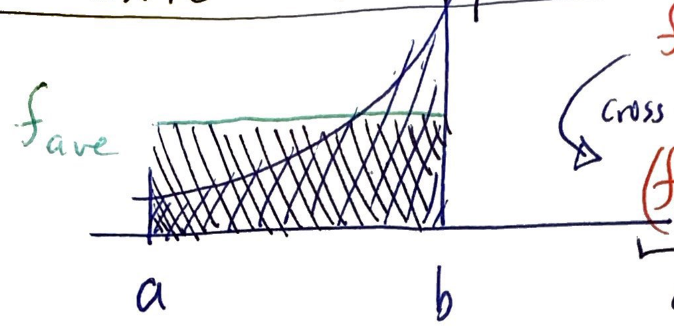
$$= \frac{1}{2} \int_{u=5}^{u=1} u^{-2} \left(\frac{du}{-2} \right)$$

$$= \frac{1}{2} \frac{u^{-2+1}}{-2+1} \left(\frac{1}{-2} \right) \Bigg|_5^1$$

$$= -\frac{1}{4} \left[\frac{1^{-2+1}}{-1} - \frac{5^{-2+1}}{-1} \right] = -\frac{1}{4} \left[-1 + \frac{1}{5} \right] = -\frac{1}{4} \left(-\frac{4}{5} \right)$$

$$= \boxed{\frac{1}{5}}$$

geometrical interpretation

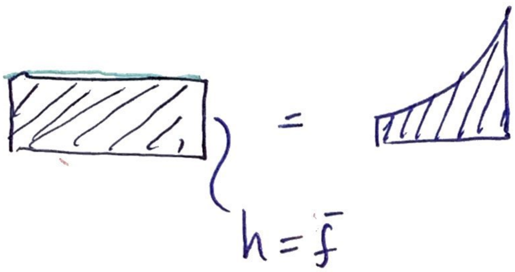


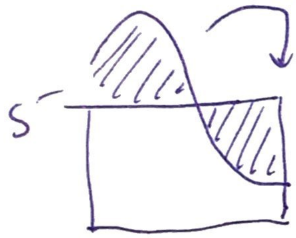
$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

cross-multiply

$$\underbrace{(f_{ave})(b-a)}_{\text{area in a rectangle}} = \underbrace{\int_a^b f(x)\,dx}_{\text{area under a curve}}$$

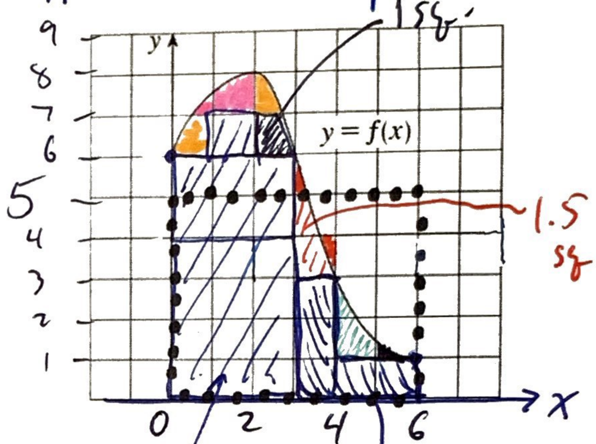
Same areas:





cut out and paste into the rectangular part.

Application: Graphical evaluation



19 squares + 5 sq. + 1

+ 1.5 sq. + 1 sq.

+ 1 sq + 1 sq = $\boxed{29.5 \text{ square units}}$

$$\bar{f} = \frac{1}{6-0}\int_0^6 f(x)\,dx$$

$$= \frac{1}{6}(29.5)$$

$$= \frac{1}{6}\cdot\frac{59}{2} = \frac{59}{12} \approx 5$$

II MVT for Integrals

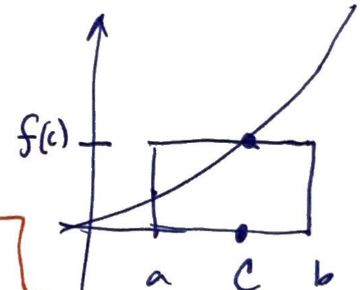
The mean value theorem for integrals:

let $f(x)$ be continous on $[a, b]$

Then there exists a number "c" such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or $$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$



Proof:

- let $F(x) = \int_a^x f(t)\,dt$, the antiderivative
- then $F(x)$ is continous on $[a,b]$
- $F(x)$ is diff'ble on (a,b)
- MVThm for derivatives (applied to F)

$$F(b) - F(a) = F'(c)(b-a)$$

- But by the Fund. Thm Calc, Part I

$$F'(x) = f(x)$$

- Therefore

$$F(b) - F(a) = f(c)\cdot(b-a)$$

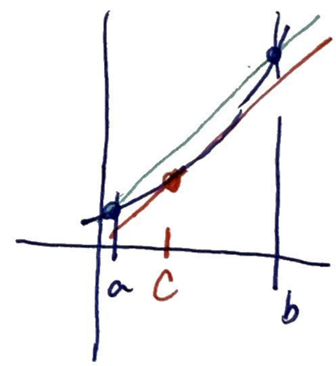
I.E. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ FTCpII

QED.

FTCpI

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$$

Recall MVT for derivatives



$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

or

$$f(b) - f(a) = f'(c)\cdot(b-a)$$

$\boxed{\text{Ex}}$ In NYC the temperature can be modelled in Fahrenheit by $T(t) = 50 + 14\sin\left(\frac{\pi}{12}t\right)$ so long as t resides between 9 am and 9 pm

Find the average temperature of the day

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let $t=0$ be 9am, let $t=12$ be 9pm

then

$$T_{ave} = \frac{1}{12-0}\int_0^{12}\left[50 + 14\sin\left(\frac{\pi}{12}t\right)\right]dt$$

$$= \frac{1}{12}\left[50t - 14\,\frac{\cos\left(\frac{\pi}{12}t\right)}{\pi/12}\right]\Bigg|_0^{12}$$

$$= \frac{1}{12}\left\{\left[50\cdot 12 - \frac{14\cdot 12}{\pi}\overbrace{\cos\left(\frac{\pi}{12}\cdot 12\right)}^{-1}\right] - \left[\cancel{50\cdot 0}^{\,0} - \frac{14\cdot 12}{\pi}\underbrace{\cos\left(\frac{\pi}{12}\cdot 0\right)}_{1}\right]\right\}$$

$$= \frac{1}{12}\left\{600 + \frac{168}{\pi} + \frac{168}{\pi}\right\}$$

$$= \frac{1}{12}\left\{600 + \frac{336}{\pi}\right\}$$

$$= \boxed{50 + \frac{28}{\pi}}$$

$$\approx \boxed{59\,^\circ F}$$

$\int \sin\left(\frac{\pi}{12}t\right)dt$

let $u = \frac{\pi}{12}t$

$du = \frac{\pi}{12}dt$

$\int \sin u\, du \left(\frac{12}{\pi}\right)$

$= -(\cos u)\cdot\frac{12}{\pi}$

$= -\frac{12}{\pi}\cos\left(\frac{\pi}{12}\cdot t\right)$