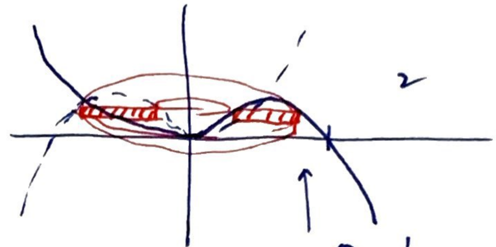
5.3 Volume by Cylindrical Shells

This technique may make some problems easier.

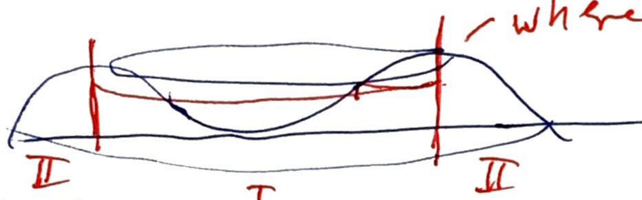
Ex

$y = 2x^2 - x^3$
rotate about y-axis



• Outer curve is the same function as inner curve.

• So break it into two regions

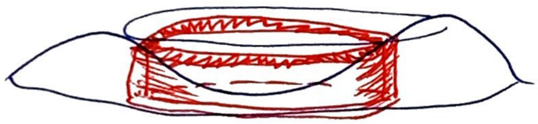


where is the peak.?

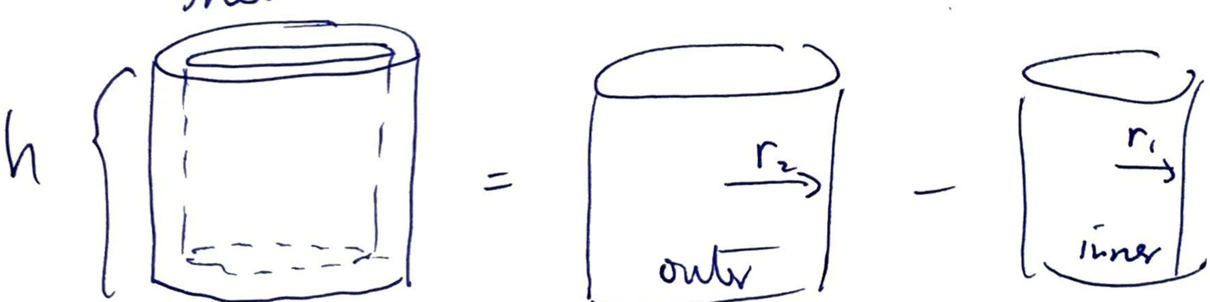
Messy!

There is a better way...
We introduce the cylindrical cylinder.

• algebraic



"shell"



$$\Delta V = V_{out} - V_{in}$$
$$= (\pi r_2^2)h - (\pi r_1^2)h$$
$$= \pi h (r_2^2 - r_1^2)$$
$$\Delta V = \pi h (r_2 + r_1)(r_2 - r_1)$$

→

$$\Delta V = \pi h (r_2+r_1)\underbrace{(r_2-r_1)}$$

$$= 2\pi h \left(\frac{r_2+r_1}{2}\right)\Delta r$$

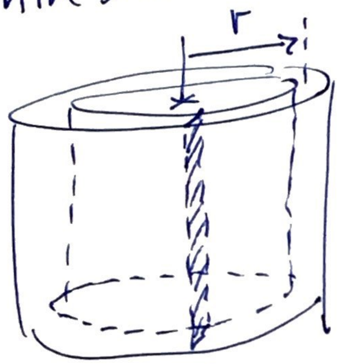
$$= 2\pi h \; r_{ave} \cdot \Delta r$$

- geometric Thin Shell

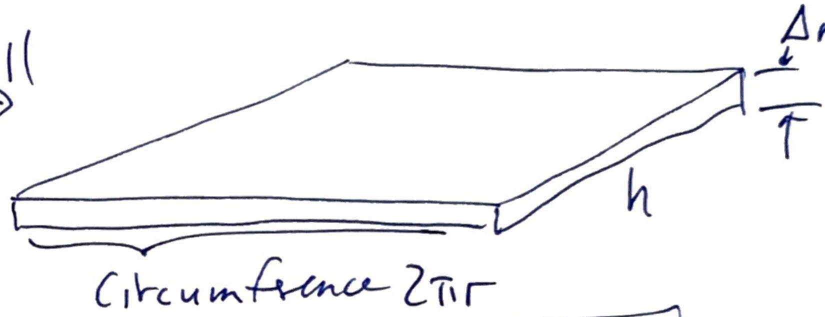
$$\boxed{\Delta V = 2\pi r \, \Delta r}$$

thickness

radius



unroll



$$\boxed{\Delta V = 2\pi r \times h \times \Delta r}$$

- Differentials approach

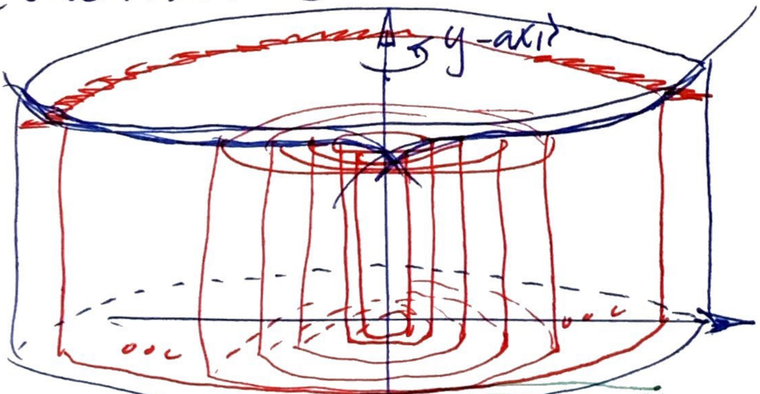
$V = \pi r^2 h$ then $\boxed{dV = 2\pi r\, dr \cdot h}$

BTW : $V = \frac{4}{3}\pi r^3 \rightarrow dV = 4\pi r^2 \cdot dr$

surface area of sphere.

- Summing up the shells ...

Use Concentric shells



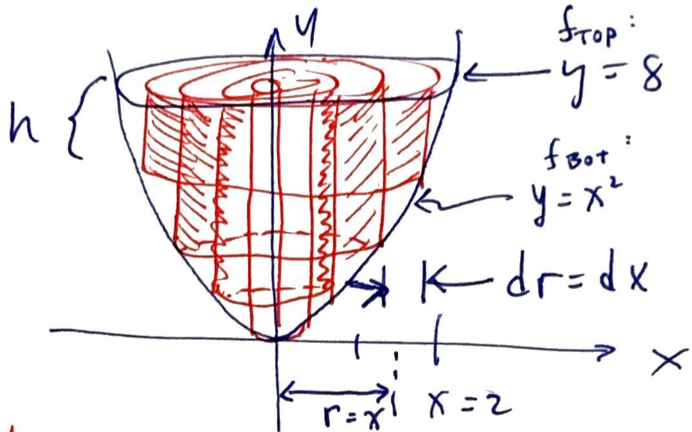
$$V \approx \sum \Delta V$$

$$\approx \sum (2\pi r) \overset{h}{f(x)} \Delta r$$

circum · height · thick

In the limit $\boxed{V = \int_{r=a}^{r=b} 2\pi r \, h(r)\, dr}$

EX



Find the volume of $y = x^2$ rotated about the y axis.

③

Concentric shells

range of x: $x = 0$ to $x = 2$

Details...

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \text{circ.} * \text{hieght} * \text{thickness}$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r \cdot h \cdot \Delta r}$$ *

insert details

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x \cdot (f_{Top} - f_{Bot}) \cdot \Delta x$$ *

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x \left[8 - x^2 \right] \Delta x$$

$$\boxed{V = \int_{x=0}^{x=2} 2\pi x (8 - x^2)\, dx}$$ *

$$= 2\pi \left[\int_0^2 8x\, dx - \int_0^2 x^3\, dx \right]$$

$$= 2\pi \left[\frac{8x^2}{2} \Big|_0^2 - \frac{x^4}{4} \Big|_0^2 \right]$$

$$= 2\pi \left[4(2^2 - 0^2) - \frac{1}{4}(2^4 - 0^4) \right]$$

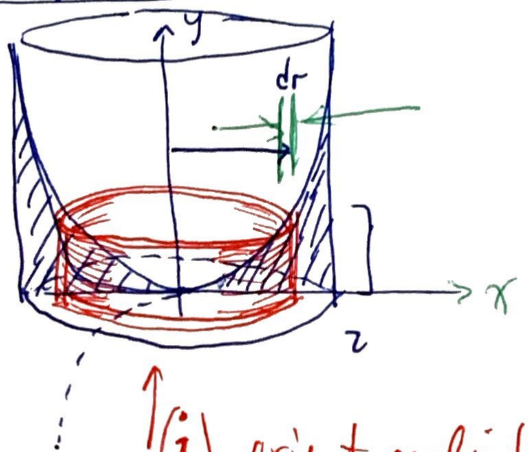
$$= 2\pi \left[4 \cdot 4 - \frac{1}{4} \cdot 16 \right]$$

$$= 2\pi \left[16 - 4 \right] = 2\pi \cdot 12 = \boxed{24\pi} \text{ cubic units}$$

EX Find the volume generated by rotating the region bound by $y = x^3$, $y = 0$ & $x = 2$ about the y-axis

(ii) geometry step

- Thickness
  $dr = dx$ ← determines integration variable
- Radius
  $r = x$
- height of cylinder
  $h = f_{TOP}(x) - f_{BOT}(x)$
  $= x^3 - 0$
  $h = x^2$



(i) orient cylinder

(iii) form the integral

$$\boxed{V = \int_{r=a}^{r=b} 2\pi r h \, dr}$$

$$V = 2\pi \int_{x=0}^{2} x\left(x^2\right) dx = 2\pi \int_0^2 x^3 \, dx$$

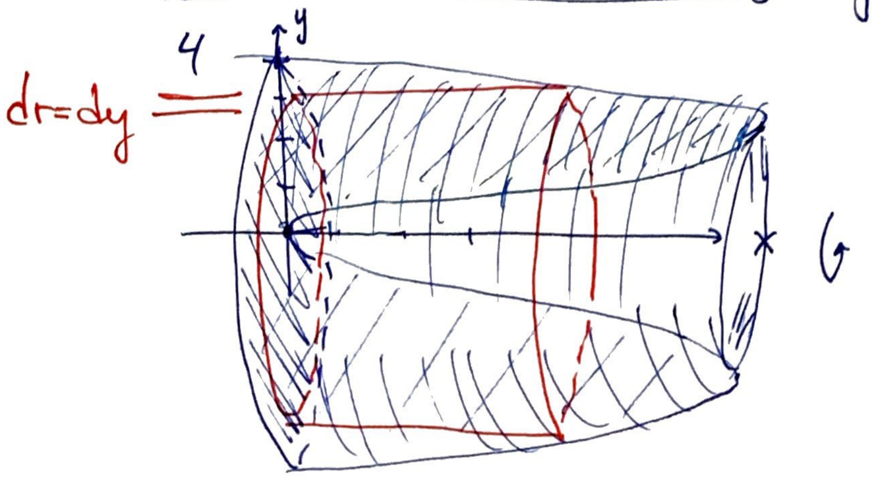
(iv) evaluate

$$V = 2\pi \left. \frac{x^4}{4} \right|_0^2$$

$$= 2\pi \left(\frac{2^4}{4} - \frac{0^4}{4} \right)$$

$\boxed{V = 8\pi}$ cubic units

Ex Rotate about the x-axis the region bounded by $x = 4y^2 - y^3$ and $x = 0$



(i) cylinder orientation

@ $x = 0$: $0 = 4y^2 - y^3$
$= y^2(4-y)$

(ii) geometry

- $dr = dy$
- $r = y$
- $h = f_r(y) - f_\ell(y)$
  $= (4y^2 - y^3) - 0$
  $= 4y^2 - y^3$

(iii) form integral

$$V = \int 2\pi r\, h\, dr$$

$$\boxed{V = 2\pi \int_{y=0}^{y=4} y(4y^2 - y^3 - 0)\, dy}$$

Set up complete

$\boxed{\text{EX}}$ Rotate the region bounded by $\Big\}$ $y = x^2$, $y = 2 - x^2$, $x = 1$ line

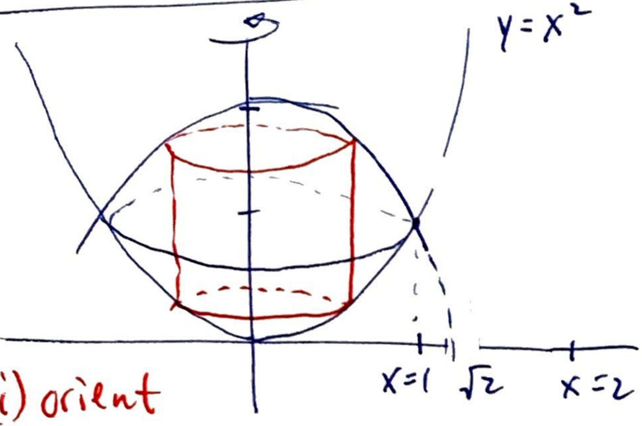
about the y-axis

(ii) geometry

- $dr = dx$
- $r = x$
- $h = f_{TOP}(x) - f_{bot}(x)$

$h = (2 - x^2) - (x^2)$

$h = 2 - 2x^2$



(i) orient

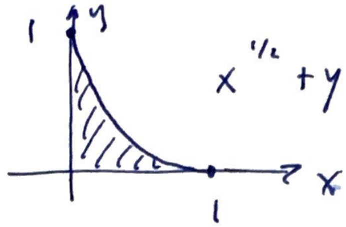
(iii) integral

$$V = \int 2\pi r h \, dr$$

$$\boxed{V = \int_{x=0}^{x=1} 2\pi x (2 - 2x^2) \, dx}$$

(iv) evaluate

EX "astroid"



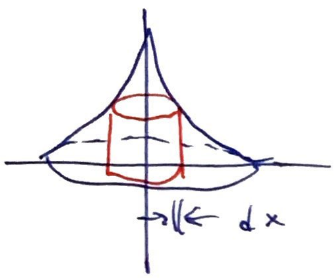
$x^{1/2} + y^{1/2} = 1$

$\to y = (1 - x^{1/2})^2$

$\to x = (1 - y^{1/2})^2$

* note that there is no asymptote
(such required chpt 7 & 8 in calc II)

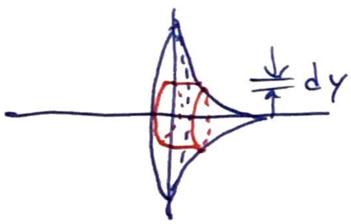
(a) rotate about y-axis



- $dr = dx$
- $r = x$
- $h = (1 - x^{1/2})^2$

$$\boxed{V = 2\pi \int_{x=0}^{x=1} x(1 - x^{1/2})^2 \, dx}$$

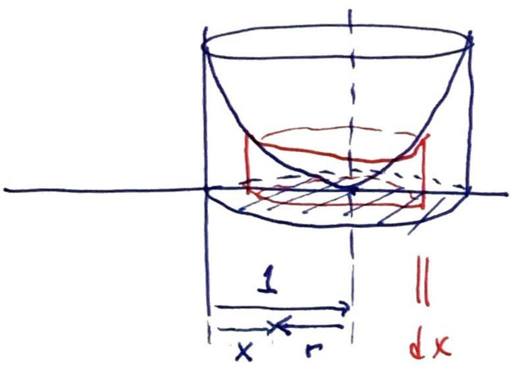
(b) rotate about x-axis



- $dr = dy$
- $r = y$
- $h = f_{right} - f_{left}$
$= (1 - y^{1/2})^2 - 0$

$$\boxed{V = 2\pi \int_{y=0}^{y=1} y(1 - y^{1/2})^2 \, dy}$$

(c) rotate about line $x = 1$

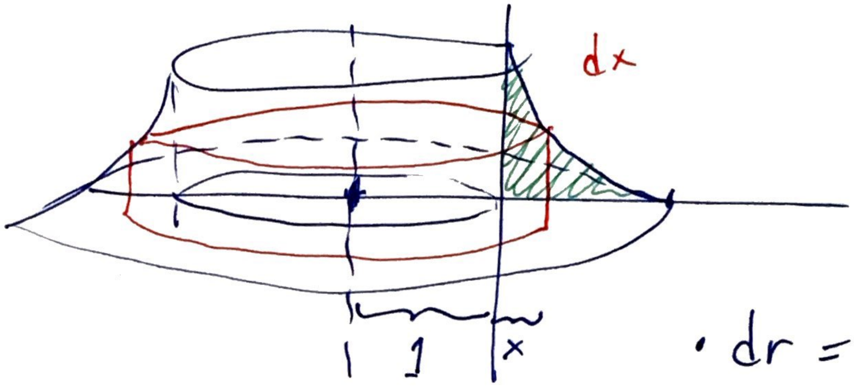


- $dr = dx$
- $r = 1 - x$
- $h = f_{Top}(x) - f_{Bot}(x)$
$= (1 - x^{1/2})^2$

radius height thick.

$$V = 2\pi \int_{x=0}^{x=1} (1 - x)(1 - x^{1/2})^2 \, dx$$

(d) rot. about $x = -1$



- $dr = dx$
- $r = 1 + x$
- $h = (1 - x^{1/2})^2$

$$\boxed{V = 2\pi \int_{x=0}^{x=1} (1+x)(1-x^{1/2})^2 \, dx}$$