$\boxed{3.9}$ **Anti derivatives** (a.k.a. Integration)

$\boxed{I}$ **Def:** A function F is called an antiderivative if, on some interval I, $F'(x) = f(x)\ \forall x \in I$

So we are going backwards!

$\boxed{Ex}$ let $f(x) = x^2$ what is it's antiderivative, i.e. F?

we seek F such that $\frac{dF}{dx} = x^2$

Ans: $F = c \cdot x^3$

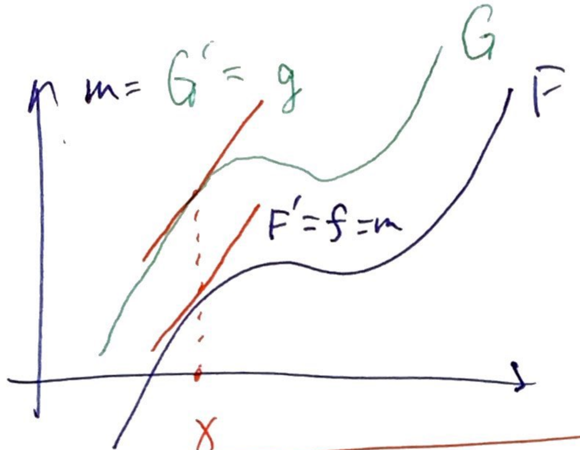
$F' = c \cdot 3x^{3-1} \equiv x^2$

so $c = \frac{1}{3}$

Final form: $\boxed{F(x) = \frac{1}{3}x^3}$

Test $\frac{d\left(\frac{1}{3}x^3\right)}{dx} = \frac{1}{3}\frac{dx^3}{dx} = \cancel{\frac{1}{3}} \cdot \cancel{3} \cdot x^{3-1} = x^2.$

* If two functions F and G have the same identical derivatives, f and g, then F and G differ by only a constant.



* The antiderivative of a function is thus only dertermined to within a constant.

So in our example: what is the antiderivatum of x^2,

we answred $\frac{1}{3}x^3$. The proper answer is

$$\boxed{\frac{1}{3}x^3 + C}$$

constant

$\boxed{\text{EX}}$ Assume $F' = f$ and $G' = g$ Find the antiderivatives of

a) $y = cf(x)$ ans: $\boxed{cF + d}$ const.

b) $y = f(x) + g(x)$ ans: $\boxed{F + G + c}$

c) $y = x^n$ where $n \neq -1$ $\xrightarrow{\text{A.D.}}$ ans: $\boxed{\frac{1}{n+1} x^{n+1} + c}$

derivative

d) $y = \cos x$ $\boxed{F = \sin x + c}$

e) $y = \sin x$ $\boxed{F = -\cos x + c}$

f) $y = \sec^2 x$ $\boxed{F = \tan x + c}$

g) $y = \sec x \tan x$

$= \dfrac{\sin x}{\cos^2 x}$

$= \sin x (\cos x)^{-2}$

diff't

$(f)^{-1}$

$[(f)^{-1}]' = -1 f^{-2} \cdot \dfrac{df}{dx}$

ans $F = (\cos x)^{-1} = \dfrac{1}{\cos x} = \underline{\underline{\sec x}}$

II Differential Equations {a preview of things to come Math212: Chpt 9}

Find f if f' is known.

EX Find f if $f'(x) = x + \frac{1}{x^3}$, let $x > 0$

This is called a differential eqn.

use $\boxed{\text{A.D. of } x^n \text{ is } \frac{x^{n+1}}{n+1}}$ x^{-3}

$$f = \frac{x^2}{2} + \frac{x^{-3+1}}{-3+1} + c$$

$$\boxed{f(x) = \frac{x^2}{2} - \frac{1}{2x^2} + c}$$

"Initial Conditions": if $f(1) = 6$ what is "c"?

$$f(1) = \frac{1^2}{2} - \frac{1}{2 \cdot 1^2} + c$$

$$6 = \frac{1}{2} - \frac{1}{2} + c$$

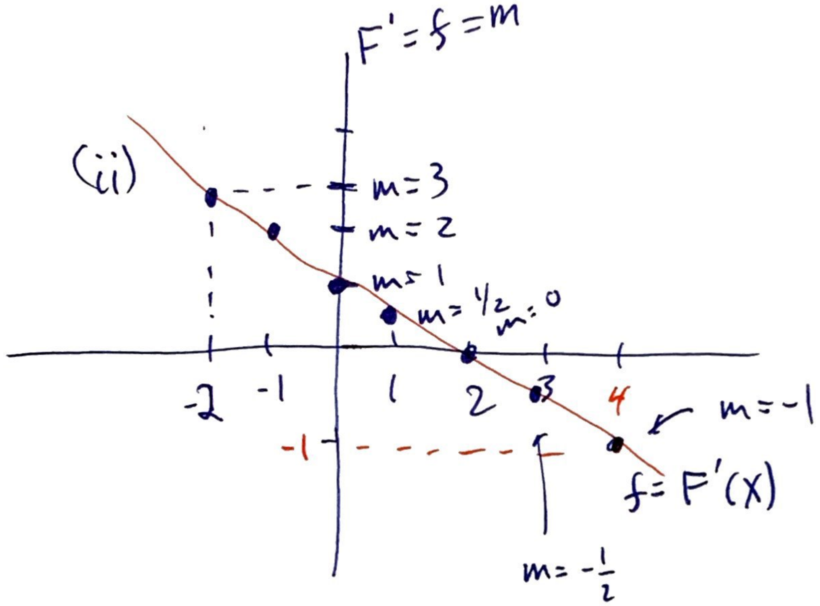
$$\boxed{6 = c}$$

Final Answer

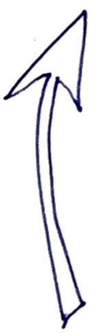
$$\boxed{f(x) = \frac{x^2}{2} - \frac{1}{2x^2} + 6}$$ is the

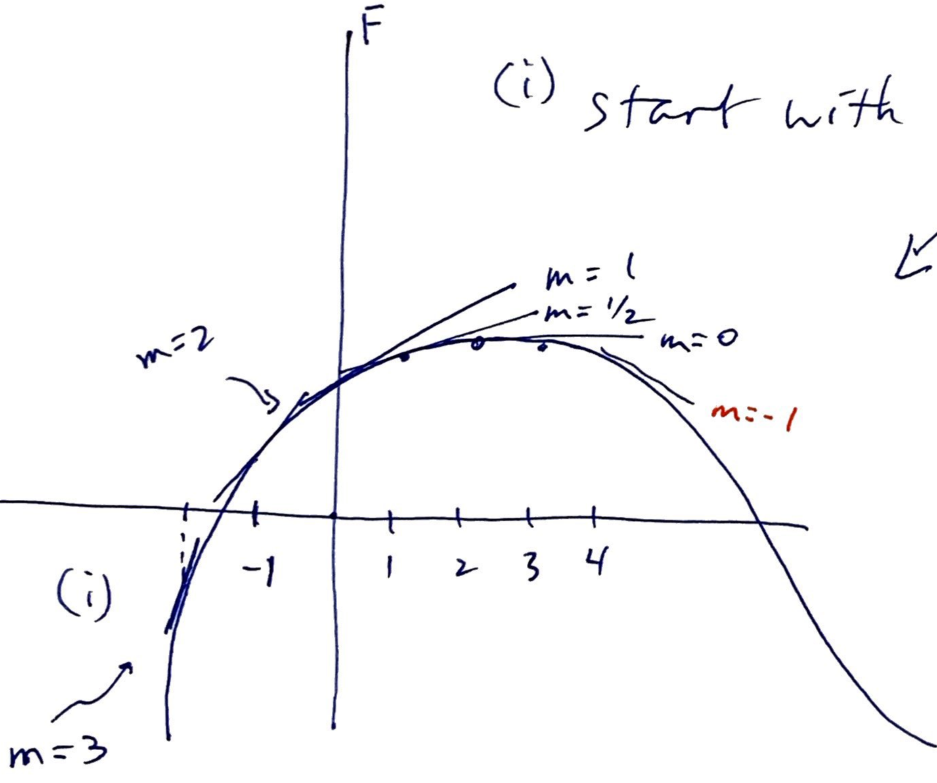
Solution to the eqn: $f' = x + \frac{1}{x^3}$ with $f(1) = 6$

⊛ Recall we went "forward" in a previous section



Given the curve below, plot the slopes in the graph above.





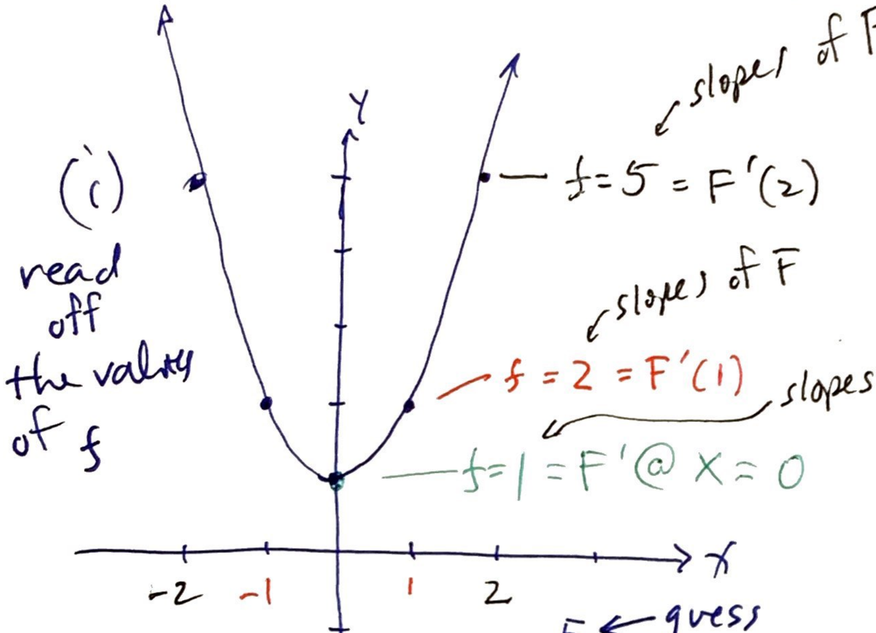
(i) start with F, (ii) plot the slopes of values F above.

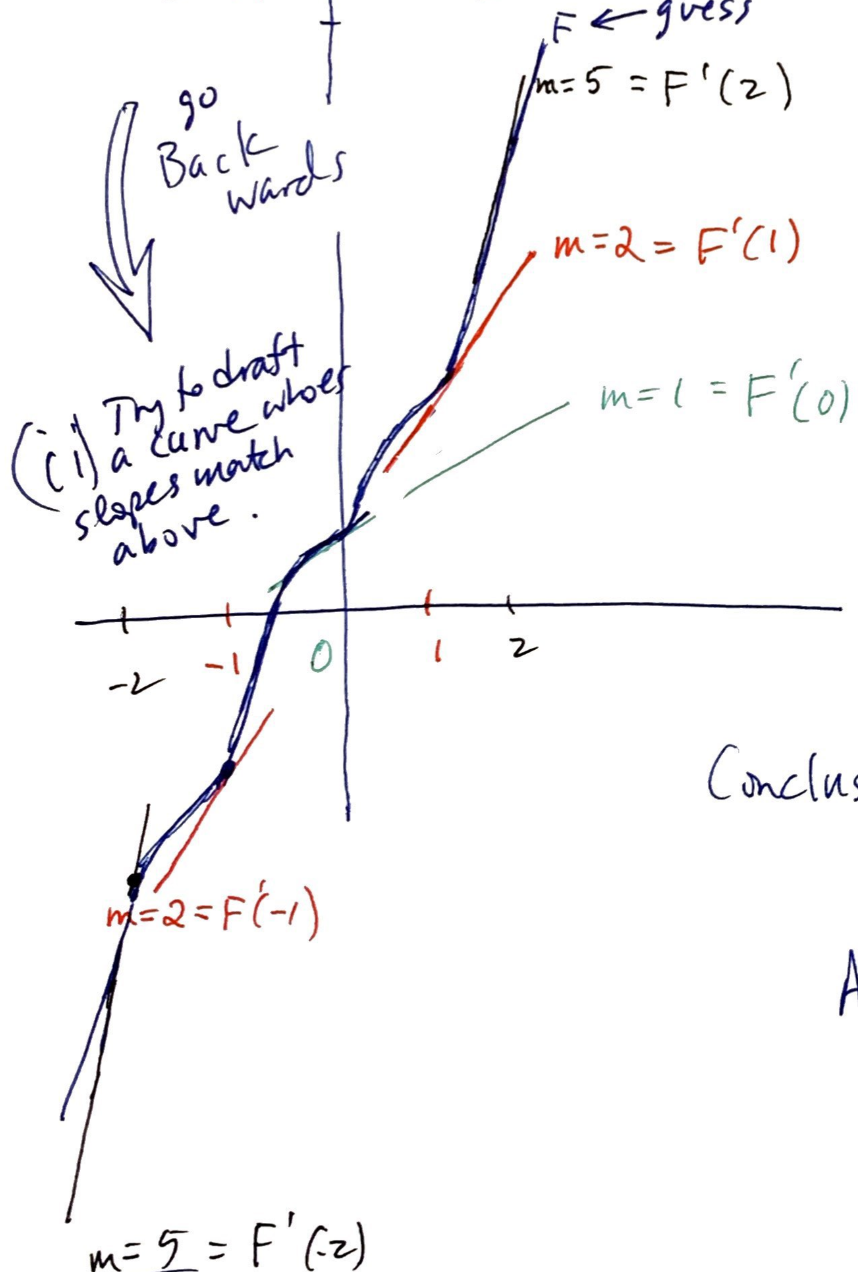
F(x) given to us

※ Now we start with a plot of the slopes …

Find a candidate F

$f(x) = x^2 + 1$





We try to find F whoes derivatives' plot matches the above

F(X) when

$\boxed{F'(X) = f(X)}$

$m = f(X)$

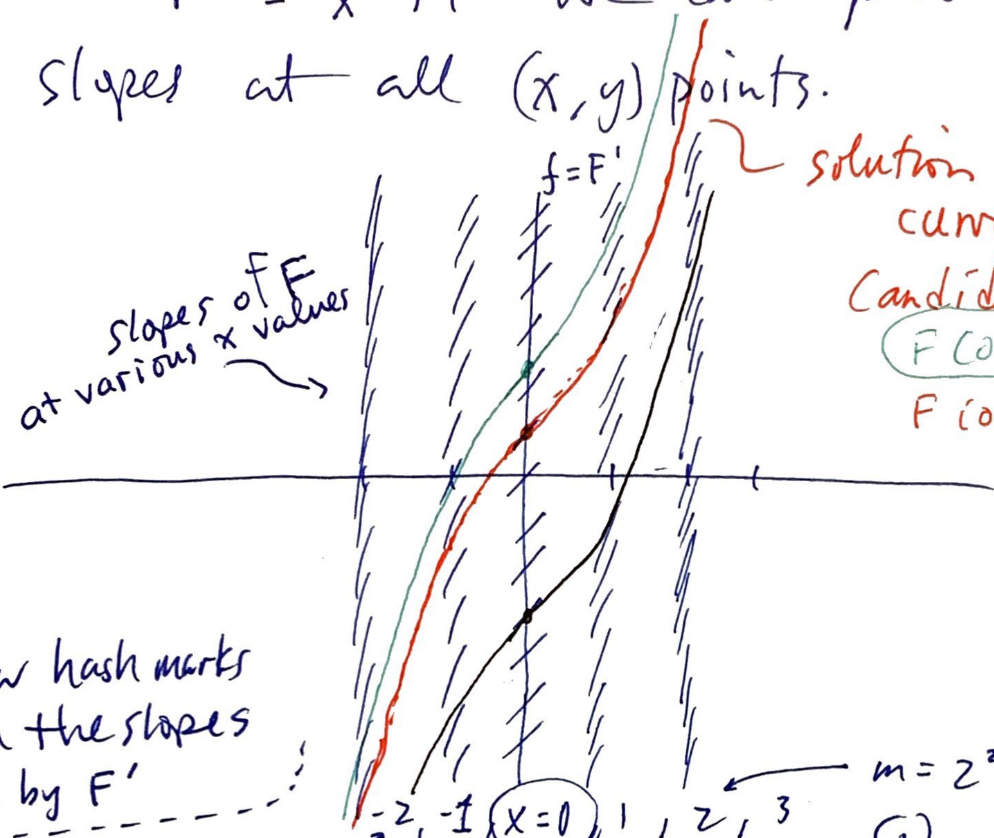
Conclusion: Painful !!

A better way →

⊛ To ease the pain we use direction fields:

given $F' = x^2+1$ (slope) we can plot "families" of slopes at all (x,y) points.

These are called "Direction fields"



solution curves: Candidate functions F.

F(0) = 2

F(0) = 1

• draw hash marks with the slopes given by F'

$m=2^2+1=5$ ③

$m=1^2+1=2$ ②

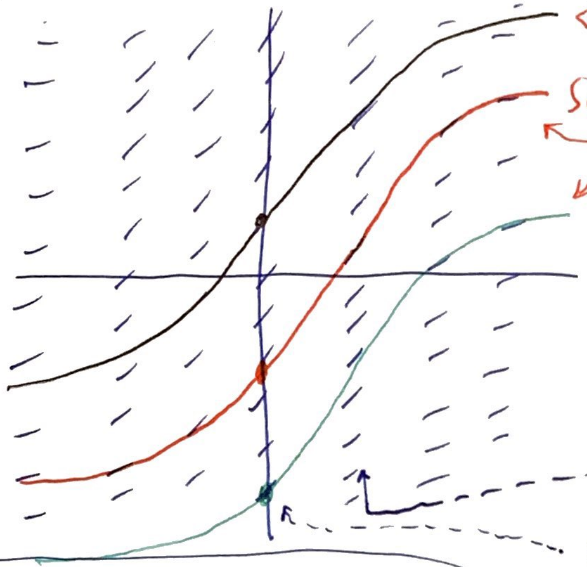
$m=(-2)^2+1 = 5$ ⑤

$m=(-1)^2+1=2$ ④

$m=0^2+1=1$ so hash mark have a slope of 45° or $m=1$ ①

do this 1st, for no particular reason.

ex Consider a direction field given to us. Plot some possible solutions.



Solution curves F

$m = \frac{1}{1+1^2} = \frac{1}{2}$

$m = \frac{1}{1+0^2} = 1$

Here $F'(x) = f(x) = \frac{1}{1+x^2}$ is the analytical eqn.

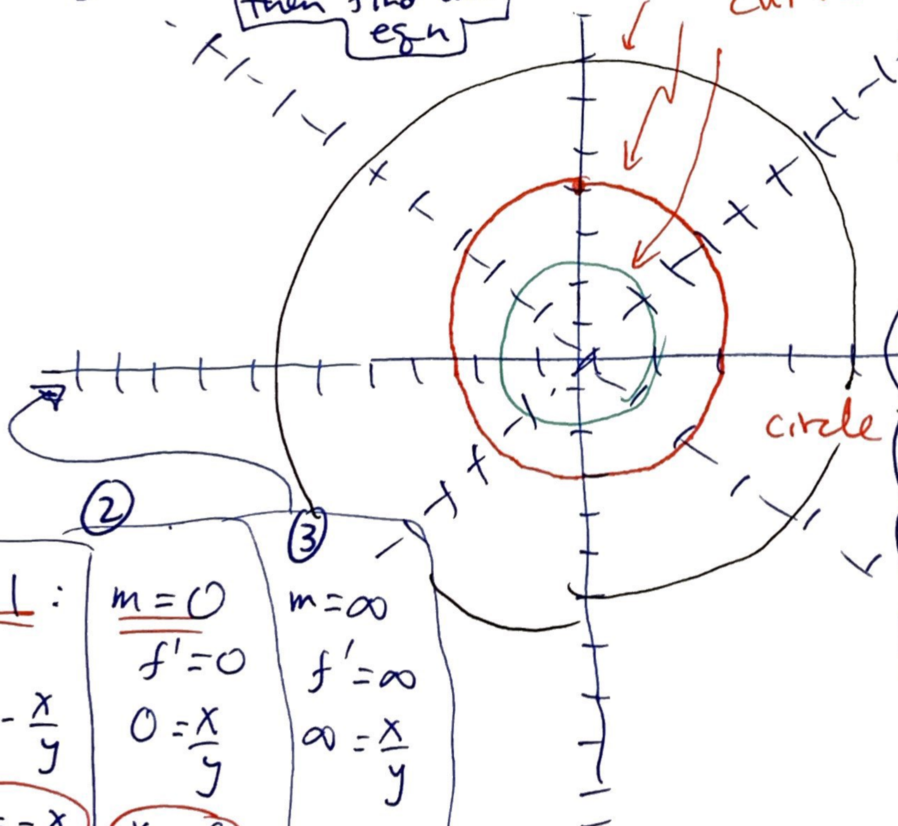
• BTW: Consider $f' = -\frac{x}{y}$, f' need not be in only x, . . . {chpt 9}

Create a direction field. Pick a slope then find an eqn

Solution curves $F(x,y)$

④ $m = -1 = f'$

$-1 = -\frac{x}{y} \Rightarrow \underline{y = x}$



They look like circles!!

Well, they are: $F(x,y)$

circle $\boxed{x^2 + y^2 = r^2}$ implicitly diff't

$2x + 2yy' = 0$

$x + yy' = 0$

$\boxed{y' = \frac{-x}{y}}$

① $m = f' = 1$: when $1 = -\frac{x}{y}$ or $y = -x$

② $m = 0$ $f' = 0$ $0 = \frac{x}{y}$ $x = 0$

③ $m = \infty$ $f' = \infty$ $\infty = \frac{x}{y}$ $y = 0$

(*) Back to analytical anti-derivatives. Lets get some more practice:

- The anti deriv. of $\cos(x)$ is $\boxed{\sin(x)}$ — A.P. / derivative
- the a.d. of $5\cos(x)$ is $\boxed{5\sin(x)}$
- the a.d. of $\cos(5x)$ is $\boxed{\frac{\sin(5x)}{5}}$
- the a.d. of $x^2\cos(x^3)$ is $\boxed{\frac{\sin(x^3)}{3}}$

  wait chain rule $(f(g))' = f' \cdot g'$
- the a.d. of $3x^2 \sin x + x^3 \cos x$,

  wait product rule??

  $$(fg)' = f'g + fg'$$

  $(x^3 \sin x)$ $\boxed{x^3 \sin x}$
- the a.d. of $\frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}$

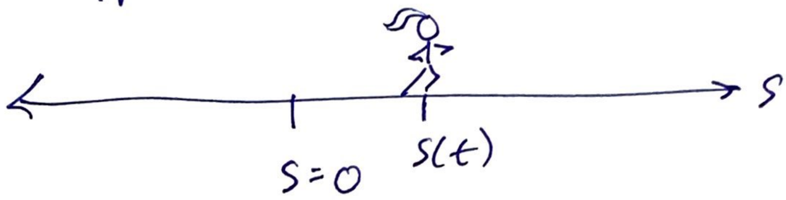
  wait, the quotient rule...

  $$(f/g)' = \frac{f'g - g'f}{g^2}$$

  $f = x^3$
  $g = \sin x$

  ans: $\boxed{\frac{x^3}{\sin x}}$

$\boxed{\text{III}}$ Application: Kinematics (physics)



1-Dim only
((Back and forth))

- let $s(t)$ be the displacement from some ref. pt. $s=0$.
- the speed of the runner is the derivative of position

$$v(t) = \frac{d\,s(t)}{dt}$$

- the acc'ln is the derivative of the velocity

$$a(t) = \frac{d\,v(t)}{dt} \qquad \left\{ \text{i.e. } a(t) = \frac{d^2 s(t)}{dt^2} \right\}$$

$\boxed{\text{Ex}}$ A car is traveling at 50 mi/hr then the brakes are fully applied, producing a decelleration of 22 ft/s².

Q: what is the stoping distance?

(i) we want $s(t)$ formula. (ii) We need t_{stop} time then (iii) $s(t_{stop})$ = answer.

(i) $a = \text{const} = -22$

but $a(t) = \frac{dv(t)}{dt}$, so the antiderivative of -22 is $v = -22t + c$

or $\boxed{v(t) = -22t + c}$

We need c — When is $v(t) = 0$? $\Rightarrow$ we need "c" 1st

- we know that $v(0) = \left(50\,\text{mi/hr}\right)\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{f}}{\text{mi}}\right) = \boxed{\frac{220}{3}\frac{\text{f}}{\text{s}}}$

$v(0) = -22 \cdot 0 + c$

$\boxed{\frac{220}{3}\,\text{ft/s} = c}$

or $\boxed{v(t) = -22t + \frac{220}{3}}$

- we need $s(t)$

$$v(t) = \frac{d\,s(t)}{dt}$$

$$-22t + \frac{220}{3} = \frac{ds}{dt} \Rightarrow \text{anti deriv.}$$

$$-22\frac{t^2}{2} + \frac{220}{3}t + C = s(t)$$

So...

$$\boxed{s(t) = -11t^2 + \frac{220}{3}t + C}$$

To determine C : let $s(0) \equiv 0$

$$-11\cdot 0^2 + \frac{220}{3}\cdot 0 + C = 0 \Rightarrow \underline{\underline{C = 0}}$$

$$\boxed{s(t) = -11t^2 + \frac{220}{3}t}$$ part (i) is complete.

answer the question now ...

(ii) we need the stopping time.

- $v(t) = 0 \Rightarrow -22t + \frac{220}{3} = 0$

$$t_{stop} = \frac{220}{3\cdot 22} \quad \text{or} \quad \boxed{\frac{10}{3}\text{ sec}}$$

(iii) plug this into the distance formula

- $s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3}\left(\frac{10}{3}\right)$

$$= \frac{1100}{9}\text{ ft}$$

$$\text{or} \approx \boxed{122.2\text{ ft}}$$