3.7 Optimization Problems (word problems! ya!) ①

I Strategy

Frequently in life, we want to know when certain values are at their max or at their min.

(X) These are Called Optimization Problems...

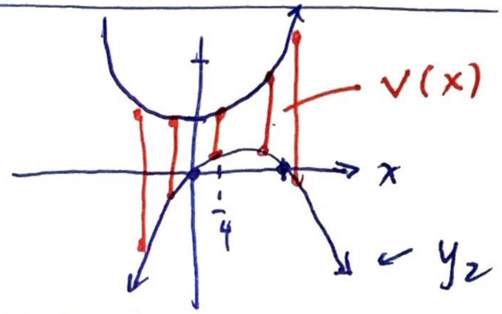
Step: 1. Understand problem
Read thouroghly

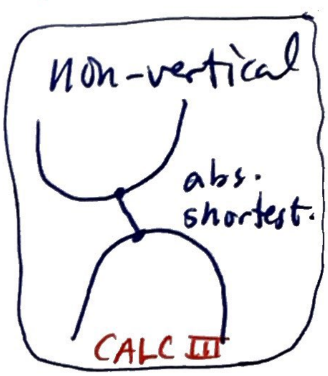
2. Diagrams
3. Introduce notation & variables that describe the quantity you seek to maximize (or minimize)
4. express the quantity you want to optimize in terms of the other variables.
5. combine variables so that only one is the independent variable.
6. Use calculus to find the extreme values and evaluate.

EX where does, and what is, the minimum vertical distance between the two parabolas

$$\begin{cases} y_1 = x^2+1 \\ y_2 = x - x^2 \end{cases}$$

- diagram



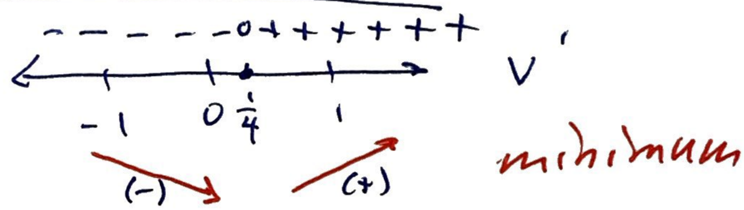


- let v = vertical distance

$$v = y_1 - y_2$$

- $\underline{\underline{v(x)}} = (x^2+1) - (x - x^2) = \underline{\underline{2x^2 - x + 1}}$

- $v' = 4x - 1$ $\Big\{$ $\underline{v' = 0 \text{ @ } x = 1/4}$



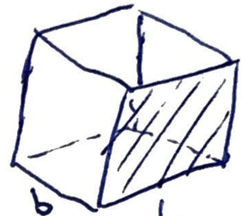
- State answer:

v is a minimum @ $\underline{\underline{x = 1/4}}$. ← where

$v\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right) + 1 = \frac{2}{16} - \frac{1}{4} + 1 = \boxed{\frac{7}{8}}$ units ← what

EX An open top box must contain 32,000 cm^3 Volume. The base must be a square. What are the dimensions that will use the least <u>material</u> (Area) (3)

- Volume $= b^2 h \rightarrow h = V_0 / b^2$



- Surface $= b^2 + 4 * hb$
- optimize S
- write S = f(one other var.) use "b"

$$S = b^2 + 4 * \boxed{\left(\frac{V_0}{b^2}\right) \cdot b}$$

→ side (lateral area) $= h \cdot b$

$$S = b^2 + 4\left(\frac{32000}{b^2}\right) \cdot b$$

$$\boxed{S(b) = b^2 + \frac{128{,}000}{b}}$$

⊛ time to optimize.

$$S' = 2b - \frac{128{,}000}{b^2}$$

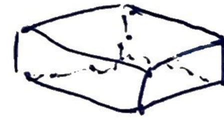
$$\frac{d\frac{1}{b}}{db} = \frac{d b^{-1}}{db} = -1 \cdot b^{-2}$$

$$S' = 0$$

$$\Rightarrow \quad 0 = 2b - \frac{128000}{b^2}$$

$$0 = 2b^3 - 128000$$

$$0 = b^3 - 64000$$



$$b = \sqrt[3]{64000} = \sqrt[3]{64}\,\sqrt[3]{1000} = 4 \cdot 10 = 40\,cm$$

- ans. question : $h = \frac{32000}{40^2} = \frac{32000}{1600} = 20\,cm$ | $\boxed{40 \times 40 \times 20\ cm}$

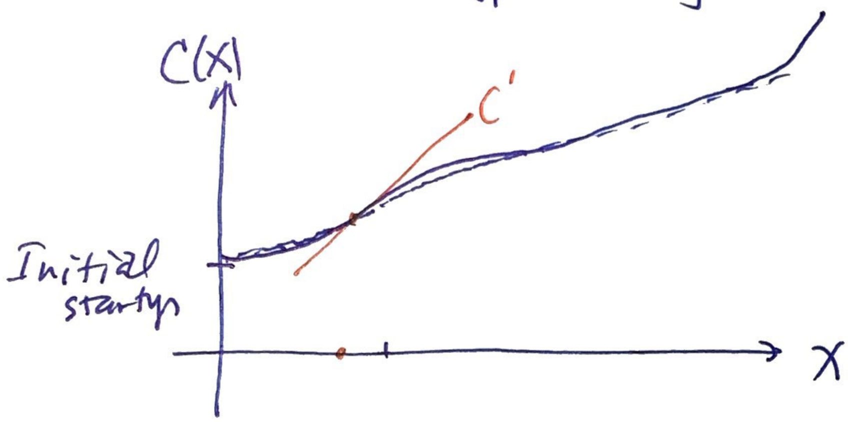
* Business applications

Definitions:

- $C(x)$ = cost function, the cost of producing x widgets.

rate of change

$C'(x)$ = marginal cost, rate of change of cost.
{ it is approximately the cost per unit $\frac{\Delta C}{\Delta x} \approx C'$



- $p(x)$ = price per unit that the company charges after selling x-units. {price function}
or demand function

- $R(x)$ = total revenue the company receives from selling x units. $R(x) = x \cdot p(x)$

$R'(x)$ = marginal revenue function

- $P(x)$ = total profit $= R(x) - C(x) = xp - C$

$P'(x)$ = marginal total profit
$c(x)$ = average cost per unit $= C(x)/x$

$\boxed{\text{EX}}$ show that if the average cost is a minimum then the marginal cost equals the average cost.

$$c = \frac{C(X)}{x}, \quad C'$$

minimum indicates we should take the derivative of $c(x)$:

$$\text{so} \quad \frac{d\,c(x)}{dx} = \frac{C' \cdot x - x' \cdot C}{x^2}$$

And at the minimum, $\frac{d\,c(x)}{dx} = 0$ so the numerator is zero.

$$\text{i.e.} \quad C' \cdot x - 1 \cdot C = 0 \Rightarrow C' = \frac{C}{x}$$

but C/x is ave cost $c(x)$: $\boxed{C'(x) = c(x)}$ @ the min of $c(x)$

$\boxed{\text{Ex}}$ If $C(x) = 16{,}000 + 200x + 4x^{3/2}$

(a) Find Cost, av. Cost and marginal cost @ 1000 widgets.

- $C(1000) = 16000 + 200(1000) + 4(1000)^{3/2}$
  $= 16000 + 200{,}000 + 40{,}000\sqrt{10} = \$342{,}491$

- $c = \dfrac{C(1000)}{1000} = \$342.49$

- $C'(x) = 0 + 200 + 4 \cdot \frac{3}{2} x^{1/2}$
  $C'(1000) = 200 + 6\sqrt{1000} = \$389.74/\text{unit}$

(b) what production level minimizes the ave. cost?

From the previous example $c'(x)$ is 0 then

$C'(x) = c(x)$

$$200 + 6x^{1/2} = \frac{16000}{x} + \frac{200x}{x} + 4\frac{x^{3/2}}{x}$$ solve for x

$$\cancel{200x} + 6x^{3/2} = 16000 + \cancel{200x} + 4x^{3/2}$$

$$2x^{3/2} = 16000 \Rightarrow x^{3/2} = 8000 \Rightarrow x = \left(\sqrt[3]{8000}\right)^2$$

$\boxed{x = 400 \text{ units}}$

So @ 400 units produced we will be running at minimum cost per unit production.

And $c(x)\Big|_{400} = \dfrac{C(400)}{400} = \dfrac{16000}{400} + 200 + 4\sqrt{400} = \$320/\text{unit}$