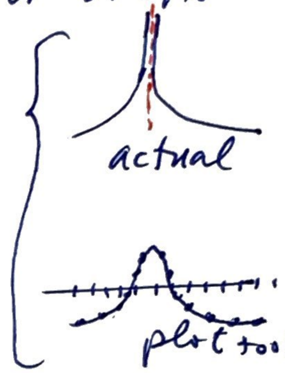
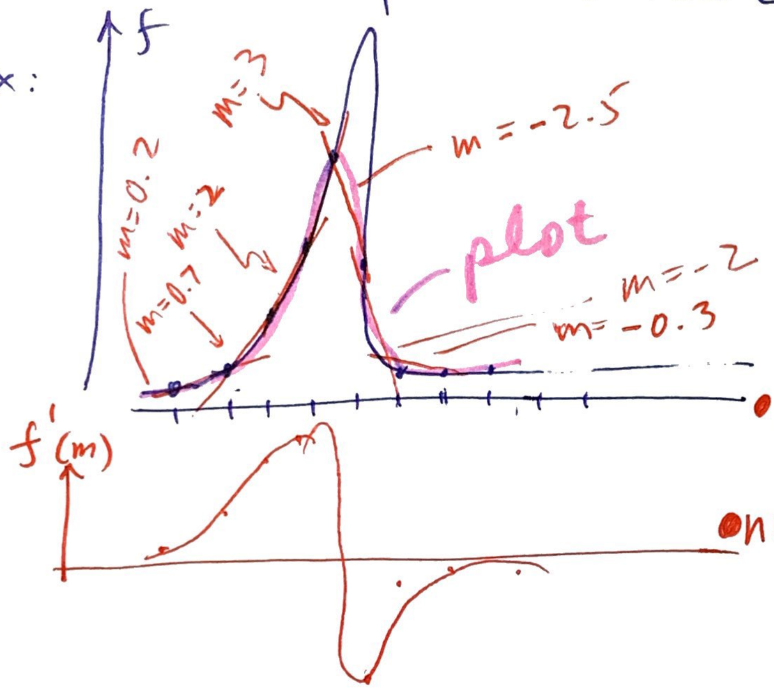
3.5 Curve Sketching

We have developed a complete set of skills to sketch curves. Plotting tools can miss features. "Dynamic Grids" will examine the numerical rate of change and add more points as needed.

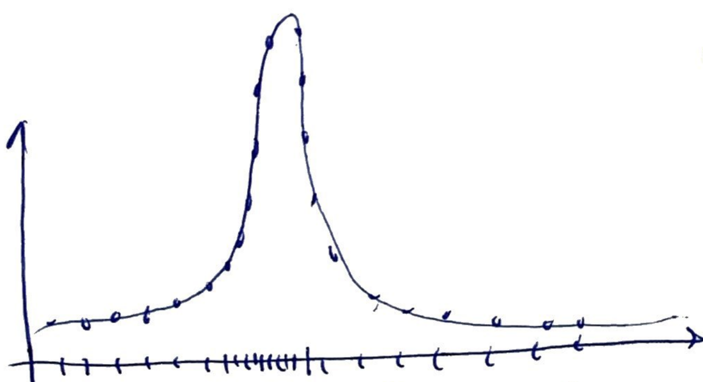


EX:



What is adaptive grid? a.k.a. Dynamic grid.

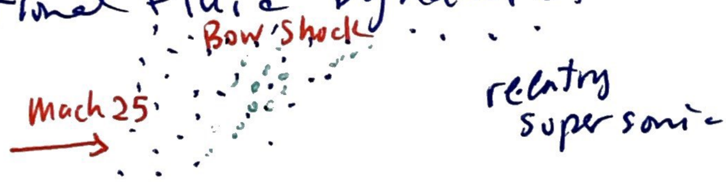
- look at fixed Δx
- numerical calculation of slopes between points.
- Dynamic grid examines changes in rate. Adds more points between rapid changes in slopes



(adaptive grids)

Add points as needed.

★ Computational Fluid Dynamics "CFD"



⊛ Tools to date

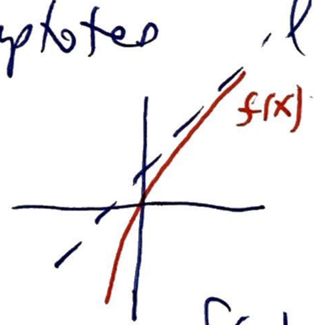
1. Domain : Find values of x where $f(x)$ is defined.

2. Intercepts:
   - $f(0)$ y-intercept
   - $f(x)=0$ solve to get x-intercepts

3. Symmetry:
   - even : $f(-x)=f(x)$
   - odd : $f(-x)=-f(x)$
   - periodic: $f(x+p)=f(x)$

4. Asymptotes:
   - $\lim_{x\to\pm\infty} f(x)=L$ L is a Horiz. Asympt.
   - $\lim_{x\to a} f(x)=\pm\infty$ a is a V.A.
   - slant/oblique asymptotes ℓ

     $\lim_{x\to\infty} f(x)-\ell(x)=0$



5. Intervals of decreasing and increasing $f(x)$.
   - $f'(x) \gtrless 0$

more . . .

6. Extrema ; critical points

- $f' = 0$ and
- f' is Not Defined (DNE)

cusp

$\Rightarrow f(c)$ evaluated.

7. Concavity and Inflection Points

- $f'' > 0$ concave Up
- $f'' < 0$ concave Down
- $f'' = 0$ at an inflection Point.

8. Helper-Points :

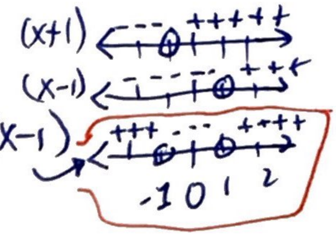
points that are easy to calculate like $f(1)$, and as needed.

- $f(x_1) = ?$
  $f(x_2) = ?$

9. Sketch the curve :

(done w/ list)

$\boxed{\text{ex}}$ Sketch $y = \dfrac{x}{\sqrt{x^2-1}} = x(x^2-1)^{-1/2}$ (4)



- Domain: x^2-1 can't be $(-)$: $x^2-1 = (x+1)(x-1)$

  $D: \underline{\underline{(-\infty,-1) \cup (1,\infty)}}$

  $\underline{\underline{\text{avoid}}}$ $\boxed{-1 \le x \le 1}$

  aka. $|x| \le 1$

- y-int: $f(0)$ but 0 is not in the domain

  x-int: $\dfrac{x}{\sqrt{x^2-1}} = 0 \Rightarrow \underline{\underline{x=0}}$ but that is not in the domain either!

  $\sqrt{x^2-1} \neq 0$ outside of $|x| \le 1$

- $f(-x)$: $\underbrace{\dfrac{-x}{\sqrt{(-x)^2-1}}}_{f(-x)} = -\underbrace{\left(\dfrac{x}{\sqrt{x^2-1}}\right)}_{f(x)}$ $\underline{\underline{\text{odd}}}$ function

- HA: $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1-\frac{1}{x^2}}} = \boxed{1}$

  $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} \underset{u=-x}{=} \lim\limits_{u\to\infty} \dfrac{-u}{\sqrt{u^2-1}} = -\lim\limits_{u\to\infty} \dfrac{u}{\sqrt{u^2-1}} = \boxed{-1}$

- VA: we $\div$ by 0 @ $\boxed{x = \pm 1}$

- $f'(x) = \left(x(x^2-1)^{-1/2}\right)' = x'(x^2-1)^{-1/2} + x\left[(x^2-1)^{-1/2}\right]'$

  $= \dfrac{1}{\sqrt{x^2-1}} + x\left[-\dfrac{1}{2}(x^2-1)^{-3/2} \cdot 2x\right]$

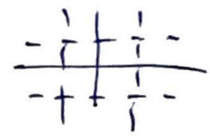
  $= \dfrac{1}{\sqrt{x^2-1}} - \dfrac{x^2}{(\sqrt{x^2-1})^3} = \dfrac{1}{\sqrt{x^2-1}} \cdot \dfrac{(\sqrt{x^2-1})^2}{(\sqrt{x^2-1})^2} - \dfrac{x^2}{(\sqrt{x^2-1})^3} = \dfrac{x^2-1-x^2}{(\sqrt{x^2-1})^3}$

  $\boxed{f' = \dfrac{-1}{(\sqrt{x^2-1})^3}}$ <0 in the domain

(cont) ↓

Ex Cont $\quad f = \frac{x}{\sqrt{x^2-1}}$, $f' = \frac{-1}{(\sqrt{x^2-1})^3}$

so $f'(x) < 0$ in $(-\infty, -1) \cup (1, \infty)$



- $f'(x)$ is not defined @ $\underline{\underline{x = \pm 1}}$ critical point. but $x = \pm 1$ not in the Domain.

  $f' \neq 0$ anywhere in the domain

  So there are no $\underline{\text{critical points}}$ in the domain.

- $f'' = \left(-(x^2-1)^{-3/2}\right)' = -\left(-\frac{3}{2}\right)(x^2-1)^{-3/2-1} \cdot (2x)$

  $= \frac{3}{\cancel{2}} \cdot \cancel{2}x \frac{1}{(x^2-1)^{5/2}} = \underline{\underline{\frac{3x}{(\sqrt{x^2-1})^5}}}$

  → $f'' < 0$ for $x < 0$, not in domain $\underline{\underline{(-\infty, -1)}}$ concave down

  → $f'' > 0$ for $x \in \underline{\underline{(1, \infty) \text{ concave up.}}}$

  → $f'' \neq 0$ No I.P. since $x = 0$ is not in the domain

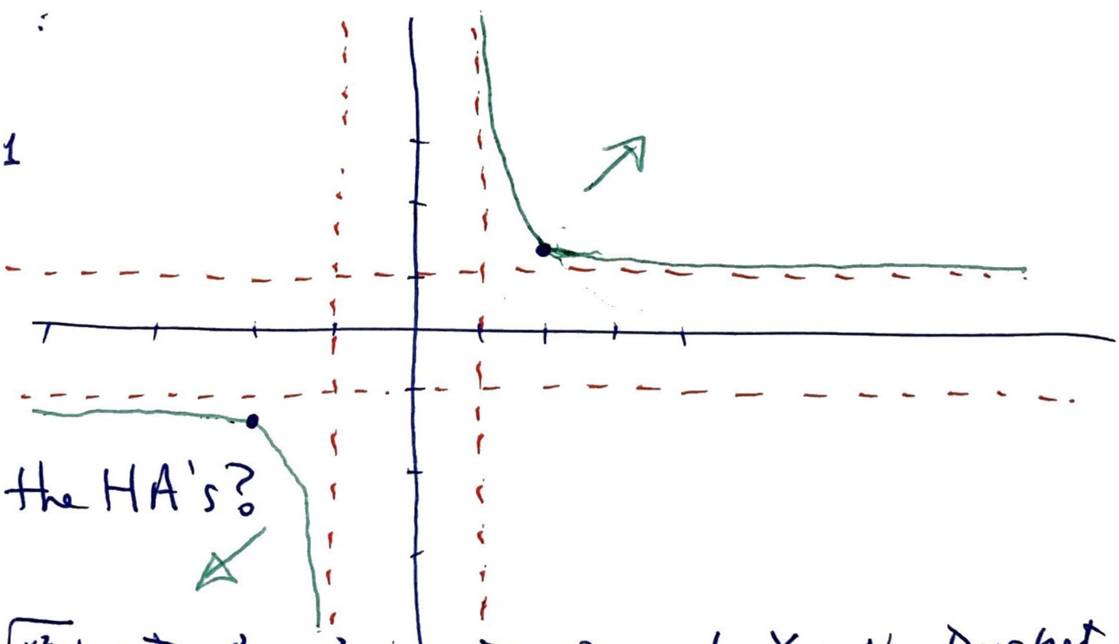
(*) Sketch it :

- Helper Points:

$f(2) = \frac{2}{\sqrt{2^2-1}} = \frac{2}{\sqrt{3}} > 1$

odd

$f(-2) = -\frac{2}{\sqrt{3}} < -1$



* Does f cross the HA's?

$f = \pm 1$

$\frac{x}{\sqrt{x^2-1}} = \pm 1 \Rightarrow x = \pm\sqrt{x^2-1} \Rightarrow x^2 = x^2 - 1 \Rightarrow 0 = -1$ * No, Does not cross HA

$\boxed{Ex}$ sketch $y = \dfrac{\ln x}{x^2}$

- Domain: $(0,\infty)$
- Int's: $f(0)$ DNE $\boxed{\text{no y-int}}$

$$\frac{\ln(x)}{x^2} = 0 \Rightarrow \ln(x) = 0 \quad \boxed{x=1}$$

- no symmetry since we do not have $(-\infty,0)$ in the domain.
- $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^2} = 0$ To show this build a table

x	$\frac{\ln x}{x^2}$
1	
10	
100	
1000	0.00000#
$\downarrow$ ∞	$\downarrow$ 0

So $\boxed{\text{HA: } y=0}$

earlier exercise in chpt 2 we did a table $\boxed{\to -\infty}$

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^2} = \lim \frac{\ln(x)}{x}\cdot\frac{1}{x} = -\infty \qquad \boxed{\text{VA @ } x=0}$$

- $$f' = \frac{(\ln x)'x^2 - (\ln x)(x^2)'}{(x^2)^2} = \frac{\frac{1}{x}\cdot x^2 - (\ln(x))(2x)}{x^4} = \underline{\underline{\frac{1-2\ln(x)}{x^3}}}$$

$\rightarrow f' > 0$ when $1-2\ln(x) > 0 \Rightarrow 1 > 2\ln(x) \Rightarrow \underline{\underline{\frac{1}{2} > \ln(x)}}$

raise both sides as exponents to base "e" $\quad e^{1/2} > e^{\ln(x)}$

$\left\{ \begin{array}{l} g(x) > f(x) \\ \text{if your base function is increasing} \\ \text{the } a^{g(x)} > a^{f(x)} \end{array} \right\}$

$e^{1/2} > x$

$0 < \boxed{x < e^{1/2}}$ increasing

$\rightarrow f' = 0$ @ $x = e^{1/2}$

$\rightarrow f' < 0$ for $x \in (e^{1/2}, \infty)$

(cont.)

- $f'' = \left(\frac{1-2\ln(x)}{x^3} \right)'$

$$= \frac{(1-2\ln x)' x^3 - (1-2\ln x)(x^3)'}{x^6}$$

$$= \frac{-\frac{2}{x}\cdot x^3 - (1-2\ln x)\, 3\cdot x^2}{x^6}$$

$$= \frac{-2x^2 - 3x^2 + 3\cdot 2\cdot x^2 \ln(x)}{x^6} = \frac{-5+6\ln(x)}{x^4}$$

$\frac{d\ln(x)}{dx} = \frac{1}{x}$

$\rightarrow f'' > 0$ when $-5+6\ln x > 0$

$6\ln x > 5$

$\ln x > 5/6$

$e^{\ln x} > e^{5/6}$

$\boxed{x > e^{5/6}}$ concave up.

$\rightarrow f'' = 0$ @ $x = e^{5/6}$ (Inflection Point)

$\rightarrow f'' < 0$ when $\boxed{0 < x < e^{5/6}}$ concave down.

$e^{1/2} \approx 1.7$

$e^{5/6} \approx 2.3$

Sketch