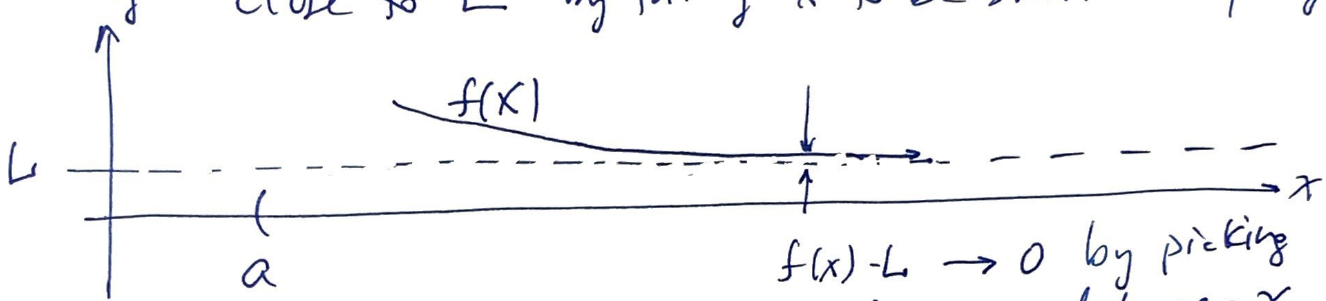
$\boxed{3.4}$ limits @ ∞ : H.A.

$\boxed{\text{I.}}$ Limits at ∞

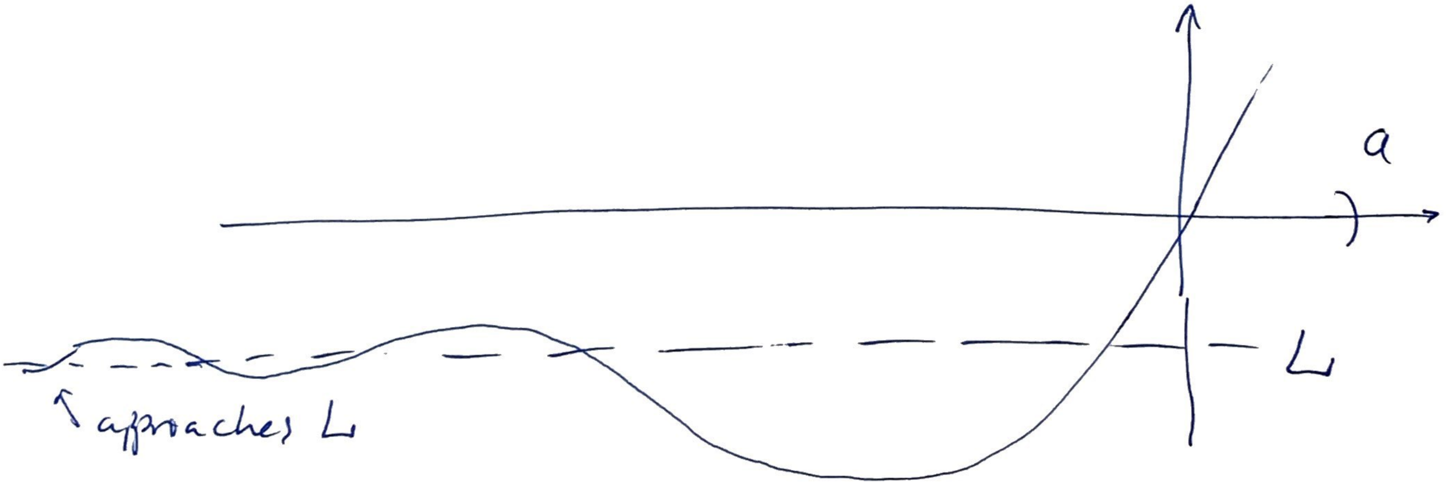
Let $f(x)$ be a <u>function</u> on (a,∞)

- Then $\boxed{\lim_{x\to\infty} f(x) = L}$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large



$f(x) - L \to 0$ by picking larger and larger x.

- Then $\boxed{\lim_{x\to-\infty} f(x) = L}$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large negative.



- The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ and/or $\lim_{x\to-\infty} f(x) = L$.

- Most limit laws we saw for regular limits also hold for limits at infinity

let $\lim_{x\to\infty} f(x) = F$ and $\lim_{x\to\infty} g(x) = G$, i.e. both limits exist.

Then a) $\lim_{x\to\infty} [f(x) \pm g(x)] = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x) = F \pm G$

b) $\lim_{x\to\infty} cf(x) = c \lim_{x\to\infty} f(x) = cF$

c) $\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x) = F \cdot G$

d) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{F}{G}$ if $G \neq 0$

e) $\lim_{x\to\infty} [f(x)]^n = \left[\lim_{x\to\infty} f(x)\right]^n = F^n$

f) $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)} = \sqrt[n]{F}$

- The same rules hold if we replace $+\infty$ with $-\infty$

- Comparison functions

(a) $\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$ and a rational number

(b) $\lim_{x\to-\infty} \frac{1}{x^r} = 0$ for $r > 0$ rational num

$\boxed{\text{EX}}$ Evaluate $\lim_{x\to\infty}\left(\frac{4x^3+3x+1}{7x^3+2x^2+x}\right) = \frac{\lim(Top)}{\lim(Bot)}$ since the Limits don't exist

Trick: divide top and bottom by x^3

$$\lim_{x\to\infty} \frac{(4x^3+3x+1)\cdot\frac{1}{x^3}}{(7x^3+2x^2+x)\cdot\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4+\frac{3}{x^2}+\frac{1}{x^3}}{7+\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 4+\frac{3}{x^2}+\frac{1}{x^3}}{\lim_{x\to\infty} 7+\frac{2}{x}+\frac{1}{x^2}} = \frac{\lim_{x\to\infty}4+\overset{\nearrow 0}{\lim_{x\to\infty}\frac{3}{x^2}}+\overset{\nearrow 0}{\lim_{x\to\infty}\frac{1}{x^3}}}{\lim_{x\to\infty}7+\underset{\searrow 0}{\lim_{x\to\infty}\frac{2}{x}}+\underset{\searrow 0}{\lim_{x\to\infty}\frac{1}{x^2}}} = \boxed{\frac{4}{7}}$$

⊛ Pre Calc:

HA:
- deg(top) < deg(bot) then $\lim_{x\to\infty}\frac{Top}{Bot} = 0$
- deg(top) = deg(bot) then $\lim_{x\to a}\frac{Top}{Bot} = \frac{A_n}{B_n}$
- deg(top) ≥ deg(bot) then $\lim_{x\to\infty}\frac{Top}{Bot} = \infty$

$= $ long division
$= Q(x)$

$$\frac{t(x)}{b(x)} \Rightarrow t(x)\overset{Q(x)}{\overline{\smash{)}\,b(x)}}$$

$\boxed{\text{EX}}$ Find the asympotic limit for

$$\lim_{x\to\infty} \frac{3x^5+2x+1}{-2x^2-4}$$

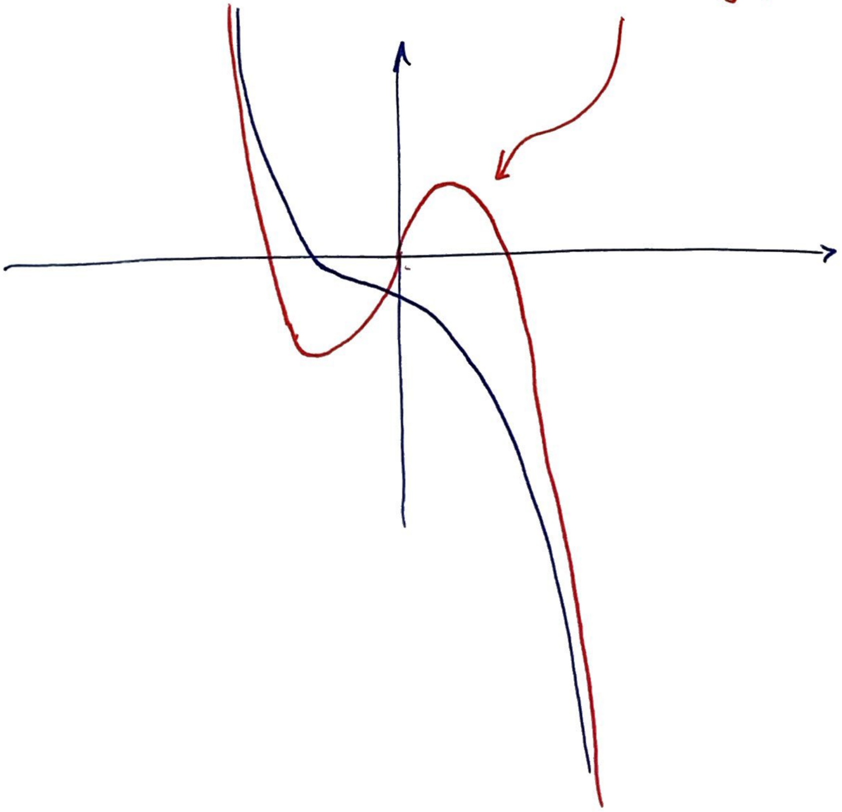
deg(top) > deg(bot) ⇒ long divide

$$\begin{array}{r} -\frac{3}{2}x^3+3x \\ -2x^2+0x-4 \,\big)\, \overline{3x^5+0x^4+0x^3+0x^2+2x+1} \\ -\underline{(3x^5+0x^4+6x^3)} \\ -6x^3+0x^2+2x \\ -\underline{(-6x^3+0x^2-12x)} \\ 14x+1 \end{array}$$

$$\boxed{\frac{3x^5+2x+1}{-2x^2-4} = -\frac{3}{2}x^3+3x+\frac{14x+1}{-2x^2-4}}$$

done

$$\lim_{x\to\infty} \text{"} = \underbrace{-\frac{3}{2}x^3+3x}_{\uparrow\ \text{oblique asymptote}} + 0$$



Ex Study $y = \frac{2x^2+1}{\sqrt{x^4-4}}$ at ± ∞.

- $\lim_{x\to\infty}\left(\frac{2x^2+1}{\sqrt{x^4-4}}\right)\cdot\frac{1/x^2}{1/x^2}$

$$= \lim_{x\to\infty}\left(\frac{\frac{2x^2}{x^2}+\frac{1}{x^2}}{\frac{\sqrt{x^4-4}}{x^2}}\right)$$

$$\frac{\sqrt{x^4-4}}{x^2} = \frac{\sqrt{x^4-4}}{\sqrt{x^4}}$$

$$= \lim_{x\to\infty}\left(\frac{2+1/x^2}{\sqrt{\frac{x^4}{x^4}-\frac{4}{x^4}}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2+\cancelto{0}{\frac{1}{x^2}}}{\sqrt{1-\cancelto{0}{\frac{4}{x^4}}}}\right) = \frac{2}{\sqrt{1}} = \boxed{2} \text{ HA}$$

- $\lim_{x\to-\infty}\left(\quad\right) = 2$ also since x^4 and x^2 terms swallow the (-).

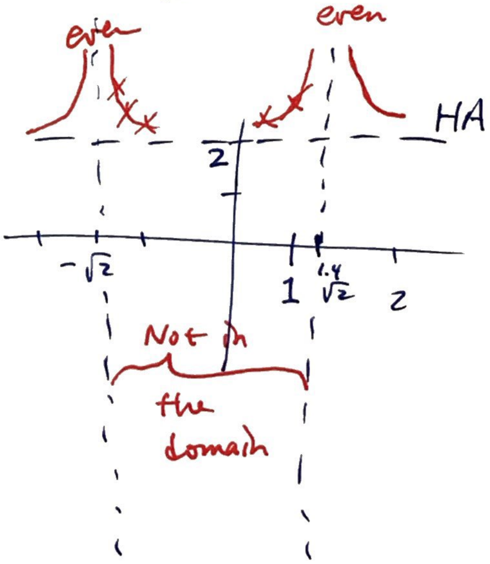
- VA? when is $\sqrt{x^4-4} = 0$?

Ans: when $x^4-4 = 0 \Rightarrow \underline{x = \sqrt{2}, -\sqrt{2}}$

$$\lim_{x\to\sqrt{2}^+}\left(\frac{2x^2+1}{\sqrt{x^4-4}}\right) = \infty$$

$$\lim_{x\to\sqrt{2}^-}\left(\frac{2x^2+1}{\sqrt{x^4-4}}\right) = \infty$$

likewise for $x \to -\sqrt{2}$



$\boxed{\text{Ex}}$ $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+2x}\right)$ — looks like $\sqrt{x^2}$ or x

---

* rationalize the numrator

$(a-b)(a+b)=a^2-b^2$

$$\lim_{x\to\infty}\left[\left(x-\sqrt{x^2+2x}\right)\cdot\frac{x+\sqrt{x^2+2x}}{x+\sqrt{x^2+2x}}\right]$$

$$=\lim_{x\to\infty}\frac{x^2-\left(\sqrt{x^2+2x}\right)^2}{x+\sqrt{x^2+2x}}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-(\cancel{x^2}+2x)}{x+\sqrt{x^2+2x}}$$

$$=\lim_{x\to\infty}\frac{-2x}{x+\sqrt{x^2+2x}}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{-\frac{2x}{x}}{\frac{x}{x}+\frac{\sqrt{x^2+2x}}{x}}$$

$$=\lim_{x\to\infty}\frac{-2}{1+\sqrt{\underbrace{\frac{x^2}{x^2}}_{1}+\underbrace{\frac{2x}{x^2}}_{0}}}=\frac{-2}{1+\sqrt{1+0}}=\frac{-2}{2}=\boxed{-1}$$

Quriosity? $\lim\limits_{x\to-\infty}\left(x-\sqrt{x^2+2x}\right)$

Chpt 11 or $\underline{\underline{2.9}}$

$\sqrt{1-x}\approx f'(a)(x-a)$ ....

$$=\lim_{x\to-\infty}\frac{-2}{1+\frac{\sqrt{x^2+2x}}{\underset{\to-\infty}{\textcircled{x}}}}=\lim_{x\to+\infty}\frac{-2}{1-1(\text{stuff})}\ \text{Not Defined.}$$

$$\lim_{x\to+\infty}\frac{-2}{1-\sqrt{1+\frac{2}{x}}}\approx\frac{-2}{1-(1-\frac{1}{2}\frac{2}{x}}=\frac{-2}{-\frac{1}{x}}=2x$$

oblique asymp.

$\boxed{\text{II}}$)

- $\lim_{x\to\infty} f(x) = \infty$ means that $f(x)$ becomes large as x becomes large.
- $\lim_{x\to -\infty} f(x) = \infty$ ditto as x becomes large negative
- $\lim_{x\to\infty} f(x) = -\infty$ means that f becomes larage neg. as x becomes large.
- $\lim_{x\to -\infty} f(x) = -\infty$ means that f becomes neg. large as becomes neg. large.

$\boxed{\text{Ex}}$ $\lim_{x\to\infty} (x^2 - x^4)$

$$= \lim_{x\to\infty} \underbrace{\cancel{x}^2}_{\infty} \underbrace{(1-\cancel{x}^2)}_{-\infty} = \underline{\underline{-\infty}}$$