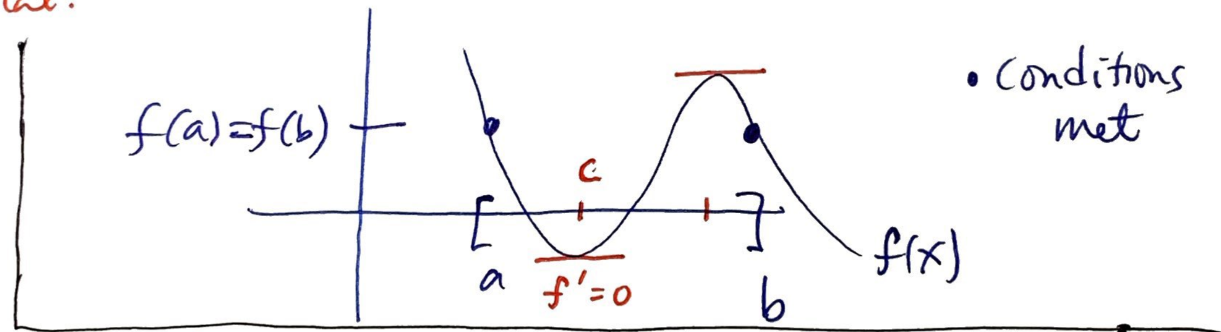
3.2 The Mean Value Thm

To get to the place where we can present the Mean Value Theorem we need some preliminary theorems.

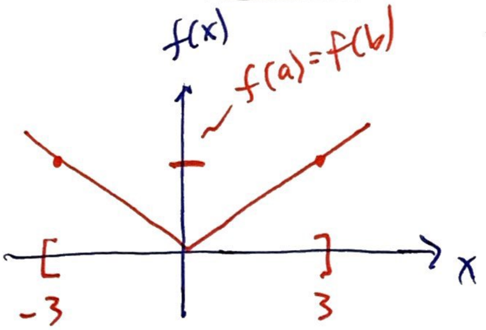
Rolle's Thm

Consider $f(x)$ such that
(i) $f(x)$ is continuous on $[a,b]$
(ii) $f(x)$ is differentiable on (a,b)
(iii) $f(a)=f(b)$ end points equal.
then there is ~~a~~ at least one number "c" in (a,b) such that $f'(c)=0$.

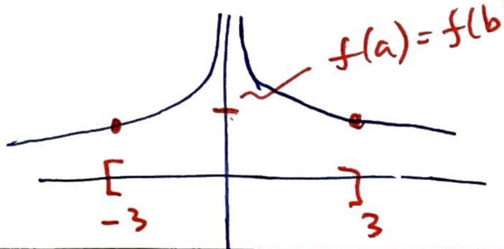
*graphical:



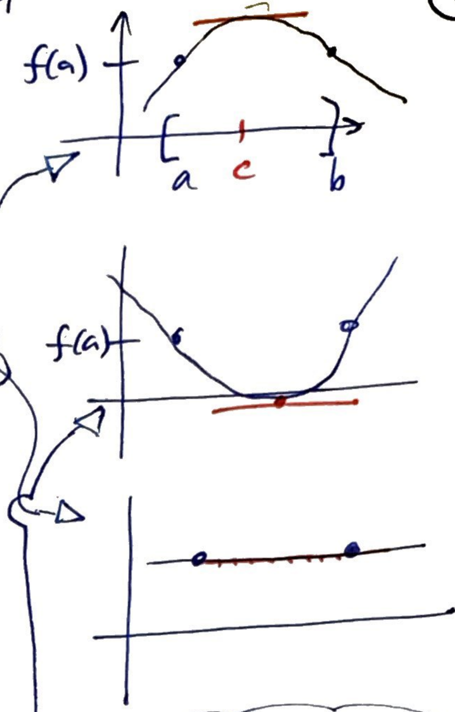
• conditions met



• fails premise # (ii) — not diff'ble so $f' \neq 0$ (anywhere for that matter)



• fails on premise (i), not continuous @ $x=0$.



Proof: Three cases

(i) If $f(x)$ is a const function then $f'(x)=0$ everywhere so any number can be taken in $[a,b]$ where $f'(x)=0$

(ii) If $f(x) > f(a)$ at some $x \in [a,b]$ then the Extreme Value Theorem says

- since $f(a)=f(b)$ the function $f(x)$ attains a max value @ $x=c$, say.
- then $f(x)$ has a local max @ $x=c$
- and since $f(x)$ is diff'ble on (a,b) Fermat's Thm says that $f'(c)=0$.

(iii) If $f(x) < f(a)$ at some $x \in [a,b]$ then the EVT says since $f(a)=f(b)$ the function $f(x)$ attains a min @c and fermat's says that $f'(c)=0$

EVT: If $f(x)$ is cont. on $[a,b]$ then $f(x)$ attains both an absolute min and an absolute max on $[a,b]$

Fermat's Thm: If $f(x)$ has a local extema @ $x=c$ and if $f'(x)$ exists @c Then $f'(x)=0$ @c.

EX Prove $x^3+2x^2+4x+8=0$ has exactly one real root. Hint: use IVT then Rolle's Thm

Let $f(x) = x^3+2x^2+4x+8$

- we poke around and find that

$\boxed{f(-3) = \underset{(-)}{-13} \quad \text{also} \quad f(-1) = \underset{(+)}{5}}$

- Intermediate value thm: there is a number "c" between $(-3,-1)$ such that $f(c)=0$, since $f(x)$ is cont.

{so we have at least one root}

– NOW –

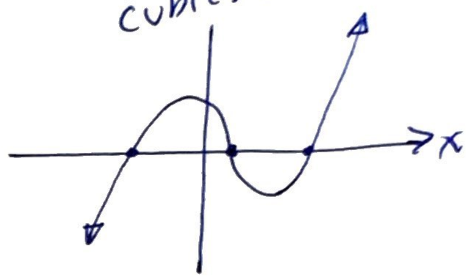
- To eliminate more than one root assume $f(x)$ has two roots @ $x=a$ and @ $x=b$. i.e, $f(a)=f(b)=0$

since $f(x)$ is a polynomial it is both continuous and diff'ble on (a,b)

Rolle's Thm says there is a number $c \in (a,b)$ where $f'(c)=0$

But $f'(x) = 3x^2+4x+4$ which is positive everywhere since this is a parabola and it has no real roots

cubics in general



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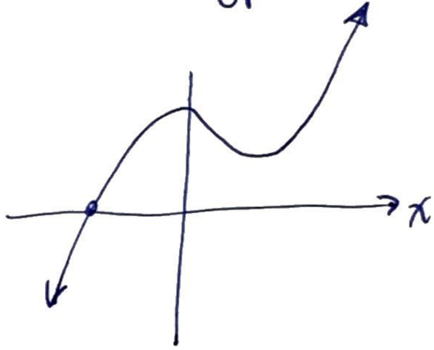

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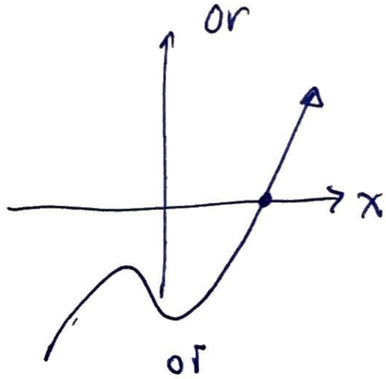

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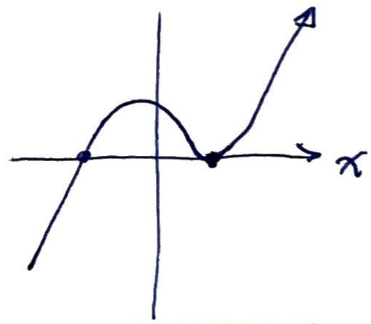
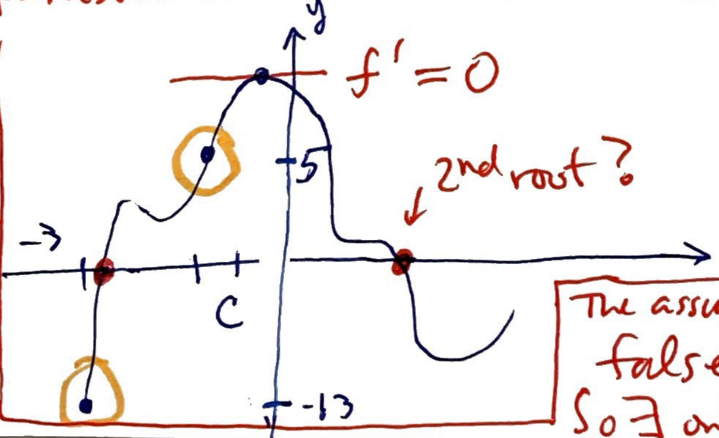


$$\frac{-4 \pm \sqrt{4^2-4\cdot3\cdot4}}{2\cdot3}$$

complex conjugates only!!

So $f' \neq 0$ anywhere

This contradicts Rolle's Thm so the premise "more than two" is not possible



The assumption is false!! So ∃ only one root

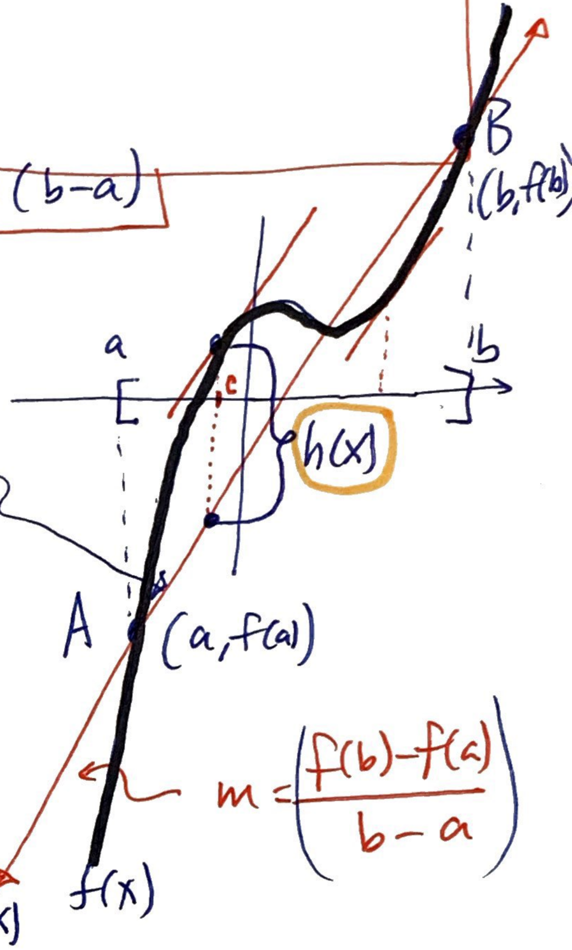
The Mean Value Thm

let $f(x)$ be both continuous and differentiable on $[a,b]$ and (a,b) respectively.

Then there is at least one number "c" in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

alternatively written as $f(b) - f(a) = f'(c)(b-a)$



Proof:

- let the secant line be

$$\boxed{y(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)}$$

- let the vertical distance function, $h(x)$ be the distance between $f(x)$ and $y(x)$

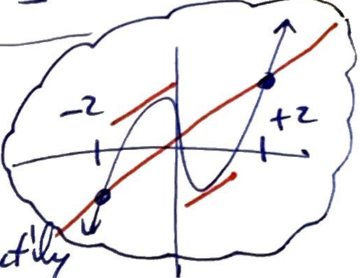
$$h(x) = f(x) - y(x)$$

$$\boxed{h(x) = f(x) - \left\{f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right\}}$$

- note $h(x)$ is cont. on $[a,b]$ since both $f(x)$ & $y(x)$ are continuous {since $f(x)$ cont. is given & $y(x)$ is a polynom.}
- note $h(x)$ is diff'ble on (a,b) because both $f(x)$ and $y(x)$ are diff'ble.
- Rolle's Thm states that there is a number $x = c \in (a,b)$ such that $h'(c) = 0$ since $\underbrace{y(a) = f(a) \text{ and } y(b) = f(b)}_{}$ — $h(a) = h(b) = 0$
- $h'(x) = f'(x) - y'(x) \overset{@c}{\Rightarrow} f'(c) = y'(c) \Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$ ■

EX Apply the mean value thm of

$$f(x) = x^3 - 3x^2 + 2 \quad \text{on} \quad [-2, 2]$$



- f is a polynomial and is therefore continuous and diff'ble on $[-2,2]$ and $(-2,2)$ respectively

- MVT says there is some number "c" such that $f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}$

$$f'(c) = \frac{-2 - (-18)}{4} = \frac{16}{4} = 4$$

LHS RHS

- but $f'(x) = 3x^2 - 6x$
  so $f'(c) = 3c^2 - 6c$

and the MVT results $\Rightarrow$ $3c^2 - 6c = 4$

- use the quadratic formula: $3c^2 - 6c - 4 = 0$

$$c = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-4)}}{2 \cdot 3}$$

$$c = \frac{6 \pm \sqrt{36 + 48}}{6} = \frac{6 \pm \sqrt{84}}{6} = \frac{6 \pm \sqrt{4 \cdot 21}}{6} = \boxed{\frac{3 \pm \sqrt{21}}{3}}$$

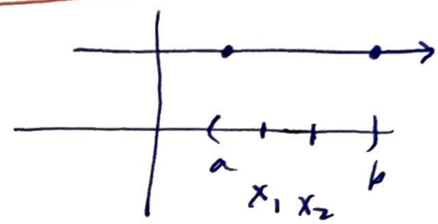
$c = 2.53$ & -0.53

keep.

$f(-0.53)$

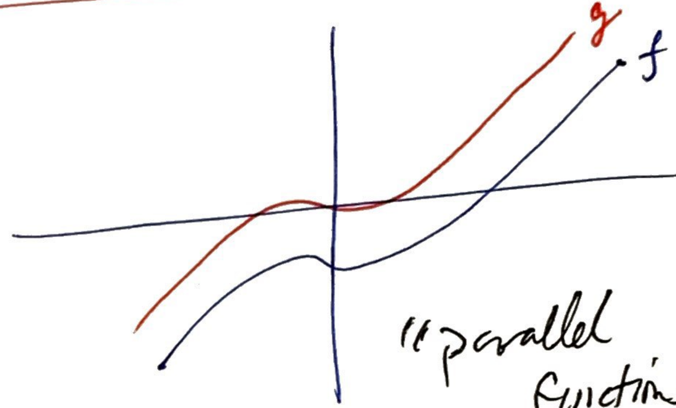
- @ $x = \dfrac{3 - \sqrt{21}}{3}$ f' is equal to 4, the slope of the secant line between end points $[-2, 2]$

Corollary : $\boxed{\text{If } f'(x) = 0 \text{ for all } x \in (a,b) \text{ then } f(x) = \text{constant function on } (a,b)}$



- proof: • let $x_1 < x_2$ be two numbers in (a,b)
  - $f'(x)$ is noted to be 0, so it exists on (a,b) then it is also cont. on $[x_1, x_2]$
  - By the MVT $\exists$ "c" so that $x_1 < c < x_2$ we have $\boxed{f(x_2) - f(x_1) = \underbrace{f'(c)}_{0}(x_2 - x_1)}$
  - But since $f'(c) = 0 \ \forall c$ then $f(x_2) - f(x_1) = 0 \Delta x$
  - So $f(x_2) = f(x_1)$: this holds for any x_1 and x_2 in (a,b)
  - $\boxed{f(x) \text{ is a constant function.}}$

Corollary: $\boxed{\text{If } f'(x) = g'(x) \text{ for all } x \in (a,b) \text{ then } f(x) \text{ and } g(x) \text{ differ by only a constant i.e. } g(x) = f(x) + k}$



"parallel functions"

Proof: let $F(x) = f(x) - g(x)$

then $F'(x) = f'(x) - g'(x)$

so we have the premise $f'(x) = g'(x)$

then $F'(x) = 0$ but by the previous corollary $F(x) = \text{const.}$

Thus $f(x) - g(x) = F(x) = \text{const.}$

So $\boxed{g(x) = f(x) + \text{const.}}$

∎