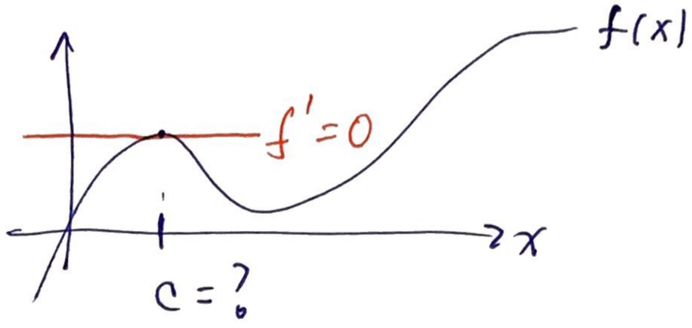
$\boxed{\text{Chapter 3}}$ Applications of Differentiation

$\boxed{3.1}$ Extrema (aka <u>optimization</u>)

Setting:



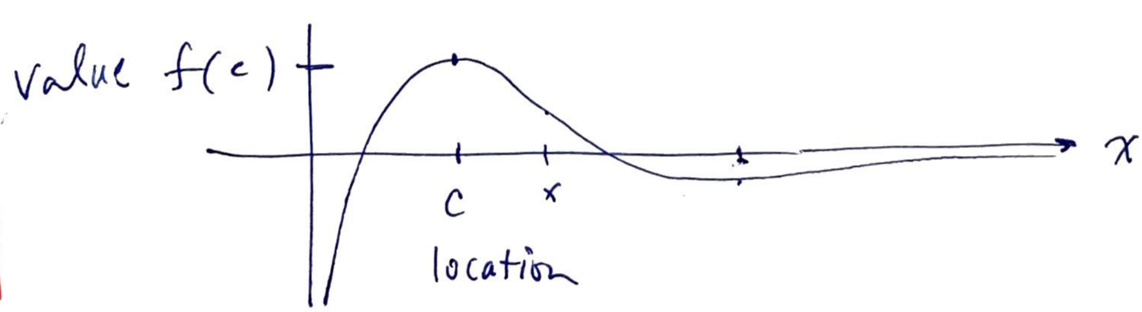
Q: given f(x), how can I find the values of x that yeild the most return from my function.

A: The extrema (maximum or a minimum) occur when the derivative of f is 0. {short answer}

DEF: "Extrema" are values of f that are maximum's or minimums.

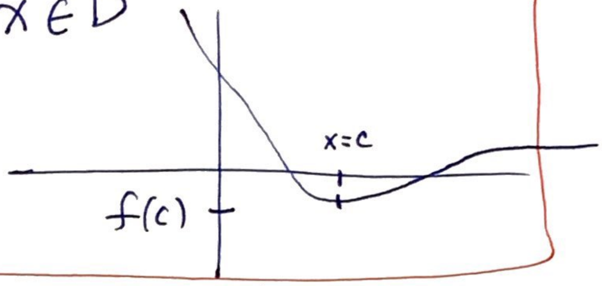
DEF: Let "c" be a number in the domain D of a function f(x).

Then f(c) is (a) The <u>absolute max</u> value of f if $f(c) \geq f(x)$ for all x in D



Then $f(c)$ is

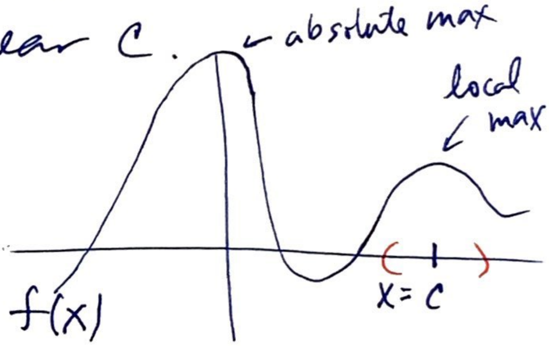
(b) the absolute minimum value of $f(x)$ is $f(c) \le f(x) \; \forall x \in D$



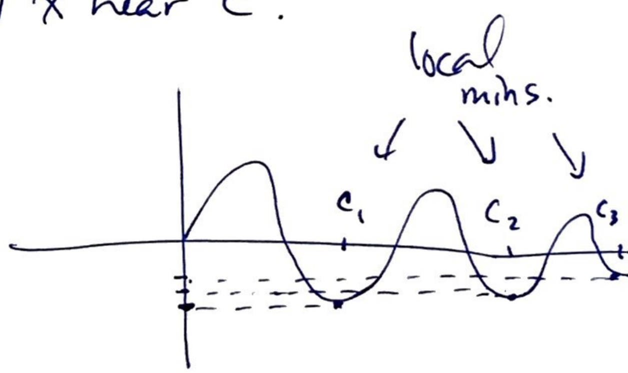
$\forall$ "forall", "$\in$" element of

DEF: The number $f(c)$ is

(a) a local maximum value of f if $f(c) \geq f(x)$ when x is near c.

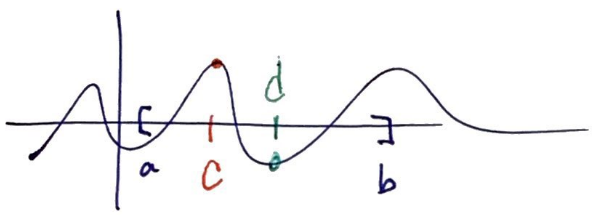


(b) a local minimum value of $f(x)$ if $f(c) \le f(x) \; \forall x$ near c.

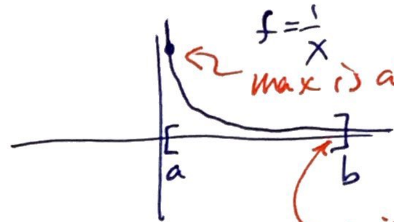


THM: The Extreme Value Theorem (EVT)

If $f(x)$ is continuous on $[a,b]$ ← closed
then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum $f(d)$ for some "c" and some "d" in $[a,b]$.



OR



*This theorem does not tell us HOW to find these extrema.

THM: Fermat's theorem

If $f(x)$ has a local extrema at $x = c$
and if $f'(c)$ exists
then $f'(c) = 0$

logic: local extrema $\underset{\text{and}}{\wedge}$ f' exists $\xrightarrow[\text{then}]{}$ $f' = 0$

logic: $a \to b$ "a" implies "b" ← conditional (if __ then __)
$b \to a$ the converse "b" implies "a"
$\sim b \to \sim a$ not "b" implies not "a" ← contrapositive

↑ negate or opposite or "not"

WARNING: Fermat's theorem does not work the other way around! (necc'y)

i.e. If $\boxed{f' = 0 \text{ @ } x=c}$, then $\boxed{f \text{ @ } x=c}$ need not be an extrema.

in other words: the converse is not necc'y true.

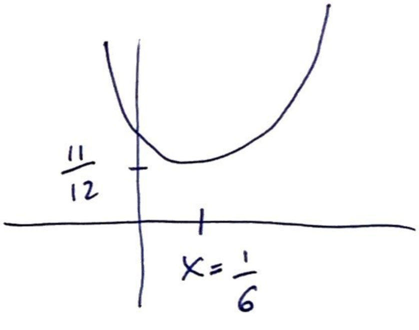
$\boxed{EX}$ Where is $f(x) = 3x^2 - x + 1$ at it's minimum value?

Fermat's Thm: $f'(x) = 6x - 1$ and set it to 0

$\Rightarrow \quad 0 = 6x - 1$

this means $x = 1/6$

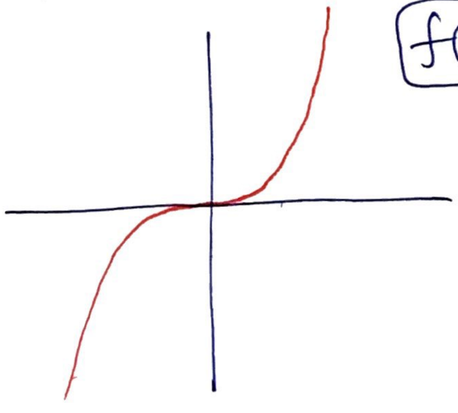
location of the minimum.



{This is the 1st time we have used a derivative to find the vertex!}

Min value: $f(\frac{1}{6}) = 3(\frac{1}{6})^2 - (\frac{1}{6}) + 1 = \frac{1}{12} - \frac{2}{12} + \frac{12}{12} = \boxed{\frac{11}{12}}$

$\boxed{EX}$ Counter example: (Local Extrema premise fails)



$\boxed{f(x) = x^3}$ $f'(x) = 3x^2$

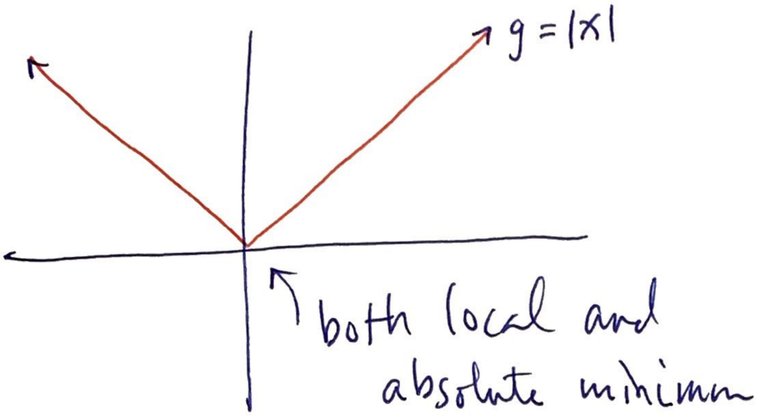
So $f'(x)$ is 0 @ $x = 0$ and f' exists.

But $f(0)$ is not a abs. max or min!

The local extrema condition was not met.

$\boxed{\text{EX}}$ Counter Example #2 (diff'ble @ minimum fails) ⑤

Verify if $g(x) = |x|$ has a local min or max @ some "c"



- $g(x)\Big|_{x=0}$ is a minimum since all other values of $g(x)$ are greater than $g(0)$.
- but g' DNE @ $x=0$
  { premise #2 fails }

So $x=0$ is a minimum but $g'(0) \neq 0$, it DNE.

(*) In order to find extreme values of $f(x)$, we look for a number "c" such that

$$f'(c) = 0 \quad \text{-OR-} \quad f'(c) \text{ D.N.E.}$$

We call these number critical points of $f(x)$.

proceedures to locate extrema in $[a, b]$ closed int'vl

(i) find all critical values of $f(x)$ in (a, b)

(ii) find the values of $f(x)$ at the endpoints "a" & "b"

(iii) { The max value in this set is the abs. max.
The min value in this set is the abs. min.

$\boxed{\text{EX}}$ Find the abs. max and min values of $f(x)$ on the interval $[0, 3]$

$$f(x) = \frac{x}{x^2 - x + 1}$$

(i) $$f'(x) = \frac{(x)'(x^2-x+1) - x(x^2-x+1)'}{(x^2-x+1)^2}$$

$$f'(x) = \frac{-x^2+1}{(x^2-x+1)^2}$$

- set equal to zero : $-x^2+1=0$ $\quad x^2 = 1$ $\quad x = +1, -1$

but only $x = 1$ is in $[0, 3]$

- $$f(1) = \frac{1}{1^2-1+1} = 1$$

(ii) endpoints

$$f(0) = \frac{0}{0^2 - 0 + 1} = 0$$

$$f(3) = \frac{3}{3^2 - 3 + 1} = \frac{3}{7}$$

critical values are

$\Rightarrow$ $\{ f(0) = 0,\ f(1) = 1,\ f(3) = \frac{3}{7} \}$

(iii) analysis of values:

abs. max on $[0, 3]$ of f is $\boxed{1}$ @ $x = 1$

abs. min on $[0, 3]$ of f is $\boxed{0}$ @ $x = 0$

BTW: when is $\underset{a=1\ \ b=-1\ \ c=1}{x^2 - x + 1 = 0}$? $\quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2 \cdot 1}$ Complex.

This means the denominator is never zero.

$\boxed{\text{EX}}$ $f(x) = \frac{1}{5}x^5 + x^4 - 4x^3 + 3$

analyze this using derivatives and desmos.com

$f'(x) = x^4 + 4x^3 - 12x^2$

$0 = x^4 + 4x^3 - 12x^2$

$= x^2(x^2 + 4x - 12)$

$= x^2(x-2)(x+6)$

$\boxed{x = 0, 2, -6}$ as critical points

$f(0) = 3$

$f(2) = \frac{2^5}{5} + 2^4 - 4(2)^3 + 3 = -33/5 = -6\frac{3}{5}$

$f(-6) = \frac{(-6)^5}{5} + (-6)^4 - 4(-6)^3 + 3 = \frac{3039}{5} = 607\frac{4}{5}$

$\frac{32}{5} + 16 - 32 + 3 = \frac{32}{5} - 16 + 3 = \frac{32 - 80 + 15}{5} = \frac{-33}{5} = -6$

$-\frac{7776}{5} + 1296 + 864 + 3 = \frac{7776}{5} + 2163\left(\frac{5}{5}\right) = \frac{3039}{5} = 607\frac{4}{5}$

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1. Find all local and absolute extrema for $f(x) = \frac{x-2}{x^2+1}$ on $[-1, 2]$

(i) critical values

(ii) $f(x)$ at end points

(iii) extrerem max =

extreme min =