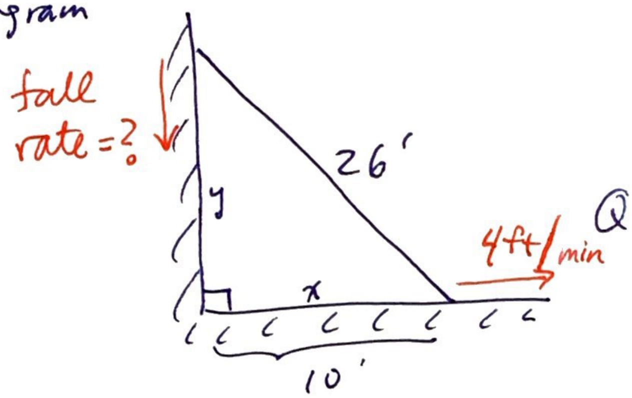
2.8 Related Rates

- In this section we look at equations and how their variables relate to each other.
- In particular we will note this relation of the variables with restpect to time

Ex Slipping ladder

(i) Diagram



If this ladder starts to slip along the floor at a rate of 4 ft/sec

Q: How quickly will the part leaning against the wall fall?

- variables: y = hieght along wall, $x_0 = \underline{10\text{ ft}}$
  x = distance out along floor, $y_0 = \sqrt{26^2 - 10^2} = \underline{\underline{24\text{ ft}}}$

(ii) Equations: Pythagorean's Thm: $\boxed{x^2 + y^2 = 26^2}$

(iii) Rates of variables w.r.t. time: Instead of dy/dx we will want dy/dt and dx/dt: Differentiate the eqn w.r.t. time

$$\Rightarrow \quad \frac{d\,(x^2+y^2)}{dt} = \frac{d\,26^2}{dt}$$

x = function of time

y = function time

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \boxed{\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}}$$

(iv) solve the problem for the wanted values:

Here we seek $\frac{dy}{dt}$ when $x = 10\,ft$ (and $y = 24\,ft$)

and $\frac{dx}{dt} = 4\,ft/min$

$\Rightarrow$ from (iii) $\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$

$$= -\left(\frac{10}{24}\right) \cdot 4\,ft/min$$

$$\boxed{\frac{dy}{dt} = -5/3\ ft/min}$$

sliding down the wall

1ft 8in per minute when $x = 10$ & $\frac{dx}{dt} = 4$.

EX The radius of a sphere is increasing at 4mm/sec

Q: How fast is the volume increasing when the radius is 80mm

(i) diagram

$r = 80\,mm$

$\frac{d\,volume}{dt} = ?$

$\frac{dr}{dt} = 4\,mm/s$

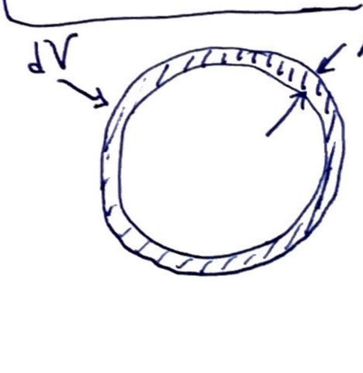
(ii) Equations

$$\boxed{V = \frac{4}{3}\pi r^3}$$

(iii) rates (diff't w.r.t. t)

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{dr^3}{dt}$$

$$\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$$

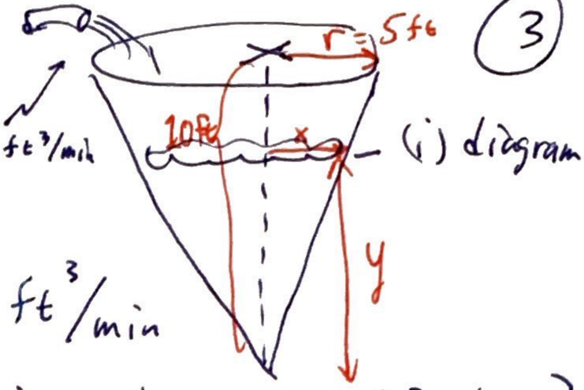


(iv) Solve for the wanted value:

$$\frac{dV}{dt} = 4\pi\,(80\,mm)^2\left(4\,\frac{mm}{sec}\right)$$

$$\boxed{\frac{dV}{dt} = 25{,}600\,\pi\ \frac{mm^3}{sec}} \left(\frac{1\,m}{1000\,mm}\right)^3 = 8 \times 10^{-5}\,m^3/sec$$

EX Inverted Conical Reservoir

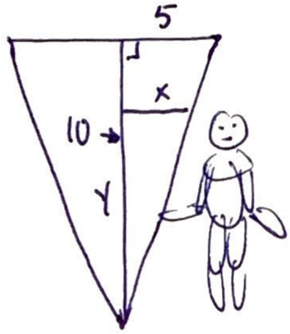


A tank is being filled at $2\ ft^3/min$

Q: How fast is the water level rising when it is 6 ft deep?

(i) diagram

Find $\frac{dy}{dt}$ when $y = 6ft$



(ii) Equation

- Similar Δ's $\frac{x}{y} = \frac{5}{10} \Rightarrow \boxed{x = y/2}$
- Volume of water $V = \frac{1}{3} A \cdot h = \boxed{\frac{1}{3}\pi x^2 y = V}$

(iii) Rates : we want to find $\frac{dV}{dt}$ and we have V, x, y. So lets eliminate one variable. We will use $y=6$ so lets eliminate "x"

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y \Rightarrow \boxed{V = \frac{\pi}{12} y^3}$$

- So $\frac{dV}{dt} = \frac{\pi}{12}\frac{dy^3}{dt} \Rightarrow \frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2 \frac{dy}{dt}$

$$\Rightarrow \boxed{\frac{dV}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}}$$

(iv) Solve : $y = 6$, $\frac{dV}{dt} = 2\ ft^3/min$

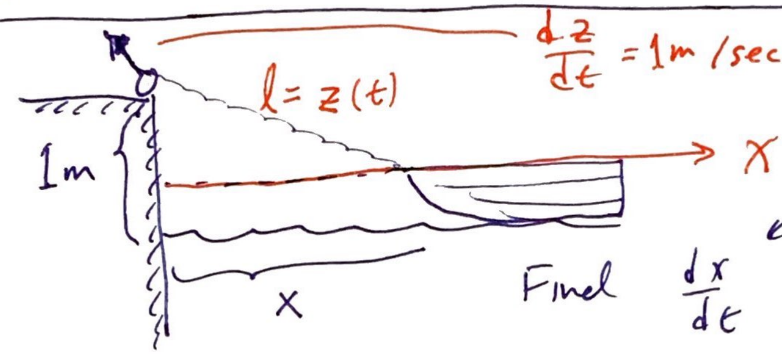
$$2 = \frac{\pi}{4}\cdot 6^2 \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2 \cdot 4}{\pi\, 36 ft^2} = \boxed{\frac{2}{9\pi}\frac{ft}{min}} \approx \underline{0.071\ ft/min}$$

EX Reeling in a boat.

A boat is being pulled into the dock by a rope passing through a pulley on the dock. The boat is 1m below the dock and is 8m out from the dock. If the rope is being reeled in at 1m/s how fast does the boat approach the dock.

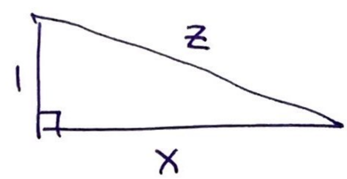
(i) diagram



speed towards dock.

Find $\frac{dx}{dt}$ when $x = 8m$

(ii) equation



$$\boxed{1^2 + x^2 = z^2}$$

(iii) rates w.r.t. time:

$$\frac{d\left(1^2 + x^2 = z^2\right)}{dt}$$

$$\Rightarrow \quad 0 + 2x\frac{dx}{dt} = 2z\frac{dz}{dt} \quad \Rightarrow \quad \boxed{\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}}$$

(iv) solve: $x = 8$, need z? $1^2 + 8^2 = z^2$ $\quad z = \sqrt{1+64} = \underline{\underline{\sqrt{65}}}$

$$\frac{dx}{dt} = \frac{\sqrt{65}}{8}(1\,m/s)$$

$$\boxed{\text{speed} = \frac{\sqrt{65}}{8}\,m/s} \approx \underline{1.01\,m/s}$$