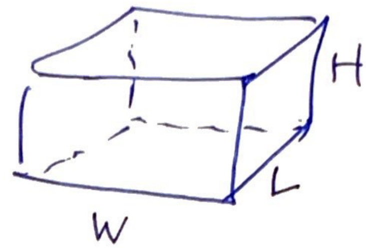
5.2 Volumes

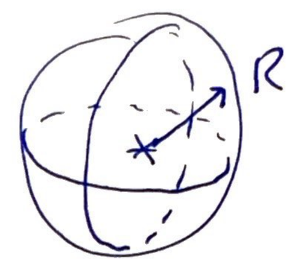
I Volumes

- Volume is 3-dimensional



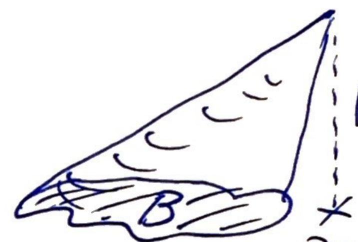
$$\boxed{V = W \cdot L \cdot H}$$

Spheres $\boxed{V = \frac{4}{3}\pi R^3}$ m^3



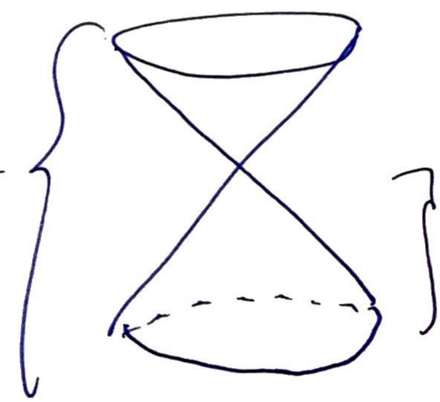
Cones $\boxed{V = \frac{1}{3} B \cdot H}$

$m^2 \cdot m$



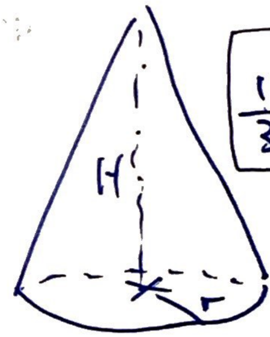
B = Base Area

Cone



$\frac{1}{2}$ Cone

Circular Right $\frac{1}{2}$ cone



$\boxed{\frac{1}{3}(\pi r^2) H}$

m^2 m

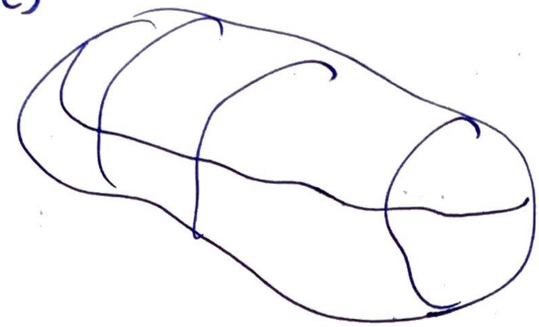
Dimensions:

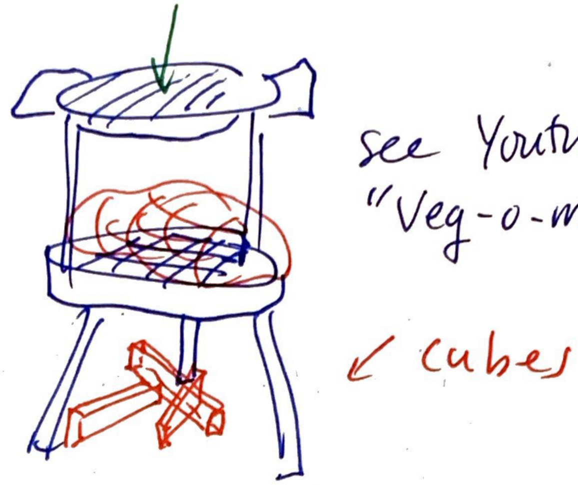
$[V] = \text{unit}^3$

(*) Computing volumes

Calc III

- Cubes



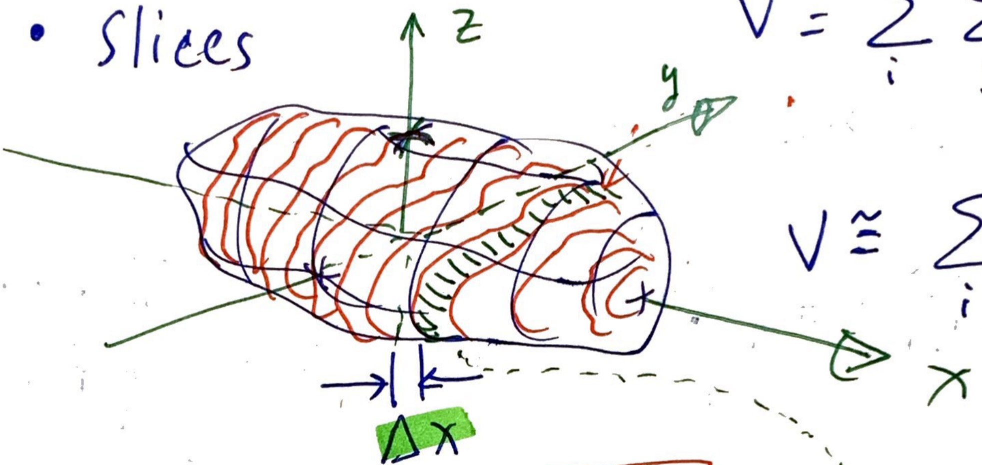


see Youtube "Veg-o-matic"

- Slices

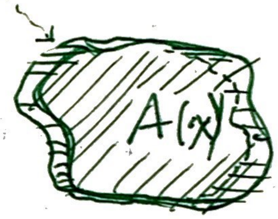
$$V = \sum_i \sum_j \sum_k v_{ijk}$$

Calc I



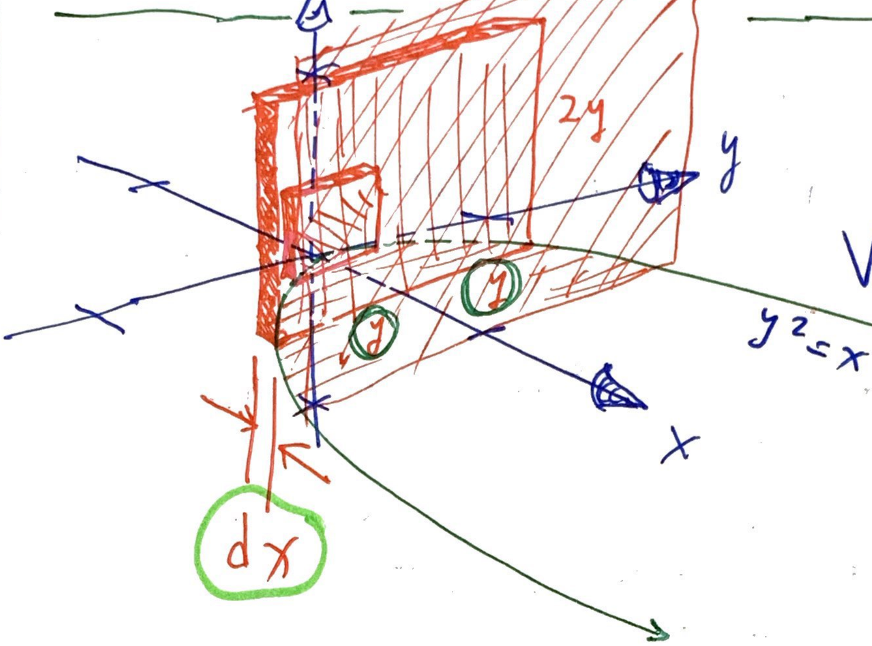
$$V \cong \sum_i (\text{Area}_i)\,\Delta x$$

$$\boxed{V = \int_a^b A(x)\,dx}$$



EX Consider a volume whose cross-sections are $\boxed{\text{squares}}$ with bottom corners attached to the parabola in the x,y plane $x = y^2$

Find the volume between $x = 0$ to $x = 1$

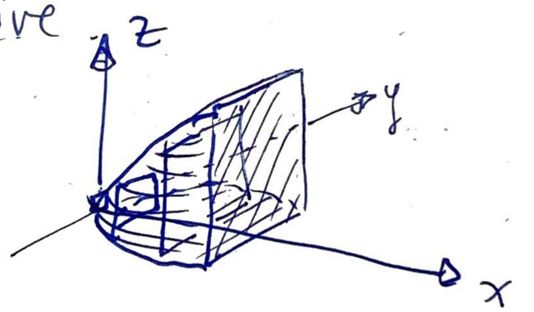


$$A(x) = (2y)^2$$

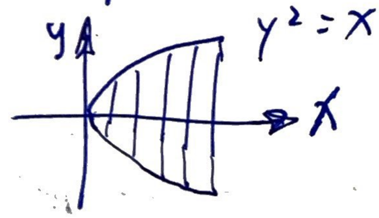
$$V = \int_{x=0}^{x=1} \underbrace{[2y(x)]^2}_{A(x)} dx$$

• side view

• perspective view



• Top view



• Volume

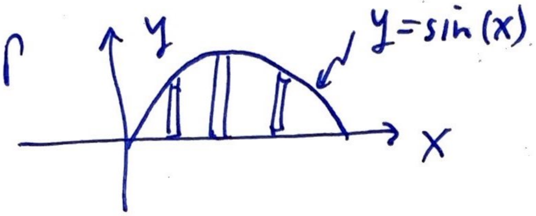
$$V = 4\int_0^1 y^2\,dx$$

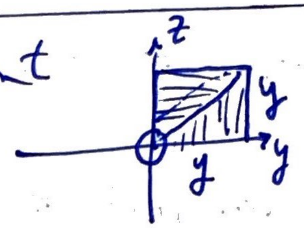
$$= 4\int_0^1 [x]\,dx$$

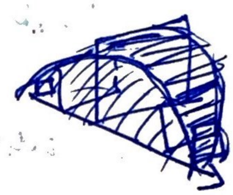
$$= 4\frac{x^2}{2}\Big|_0^1 = 2[1^2 - 0^2] = \boxed{2} \text{ cubic units}$$

math 211 5a Name ——

1. let a volume be described as one with square cross-sections whoes lower corners are attached to the x-axis and and the curve $y = \sin(x)$ {set-up only} $x = 0, \pi$

Top 

Front 



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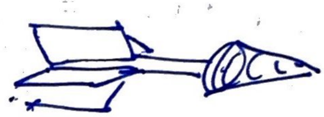
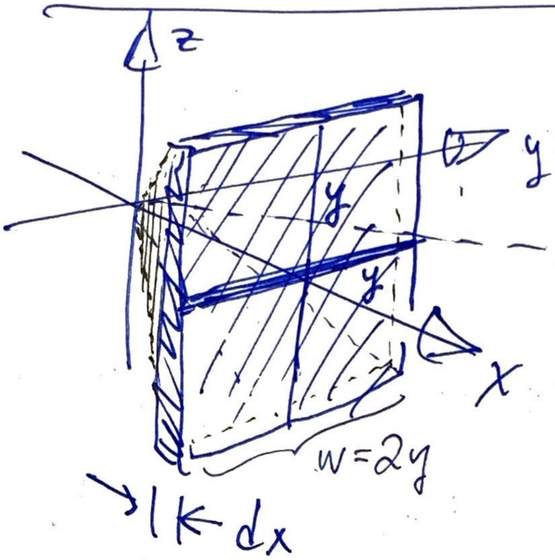
Lawn-Dart 

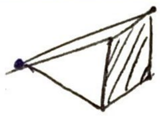
$$V = \int_a^b A(x)\,dx$$

$$A(x) = y^2 = (\sin(x))^2 = \sin^2(x)$$

$$\boxed{V = \int_0^{\pi} \sin^2(x)\,dx}$$

EX Consider a square conical shape with the vertex at the origin. The square is centered on the x-axis, the corners follow $y=x$. Set up then evaluate the volume from $x=0$ to 5





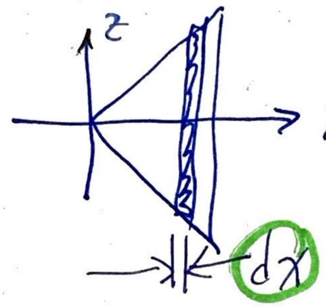
$$A(x) = w^2$$

$$w(x) = 2y$$

$$w = 2x \quad y = x$$

So $\boxed{A(x) = (2x)^2}$

• Side view

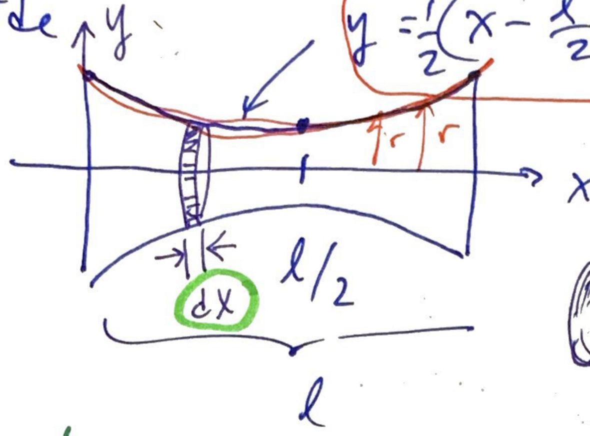


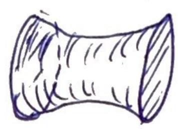
$$\boxed{V = \int_0^5 \underbrace{(2x)^2}_{A(x)}\, dx}$$

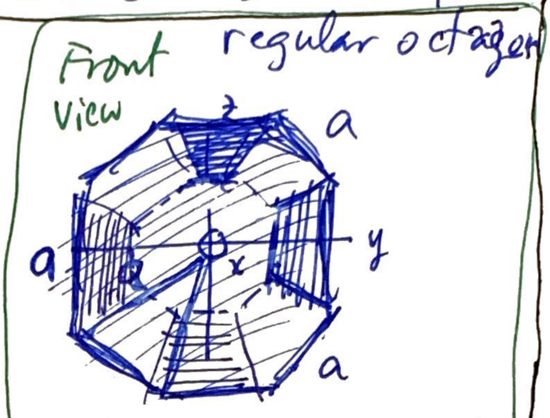
• $V = 4 \cdot \dfrac{x^3}{3}\Big|_0^5 = \dfrac{4(125)}{3} = \boxed{\dfrac{500}{3}}$ sq. units.

Ex The I-5 "National Freeway Earthquake Research Lab" uses vertical pylons that have circular cross-sections and whoes widths follow $y(x)=\frac{1}{2}\left(x-\frac{l}{2}\right)^2+\frac{l}{4}$

- Side

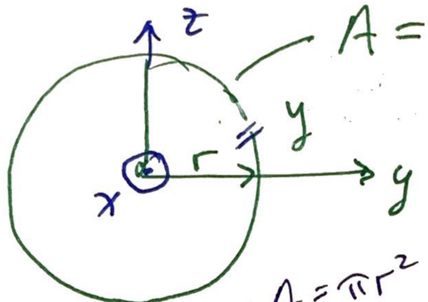






$A=2a^2(1+\sqrt{2})$

- Front view



$A=\pi r^2$

$=\pi y^2$

but $y=\frac{1}{2}\left(x-\frac{l}{2}\right)^2+\frac{l}{4}$

- So

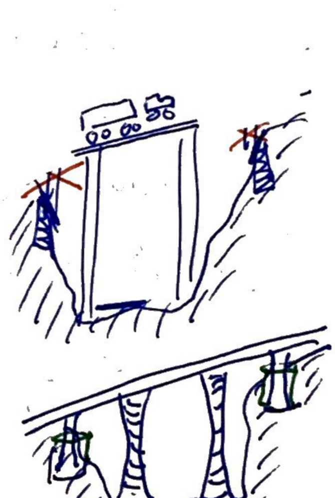
$$A(x)=\overbrace{\pi}^{A=\pi r^2}\Big[\underbrace{\frac{1}{2}\left(x-\frac{l}{2}\right)^2+\frac{l}{4}}_{r(x)}\Big]^2$$

$x=0$ to l

$$V=\int_0^l A(x)\,dx$$

$$\boxed{V=\pi\int_0^l\left[\frac{1}{2}\left(x-\frac{l}{2}\right)^2+\frac{l}{4}\right]^2dx}$$

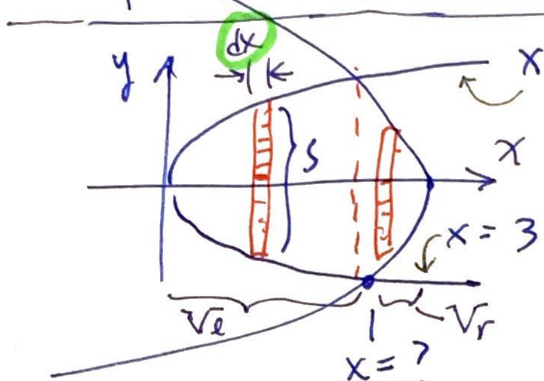
Set-up



Ex Consider the region between two parabolas

$$y^2 = x \quad , \quad x = 3 - 2y^2$$

Find the volume formed by taking equilateral triangles and pinning their corners to these parabolas.



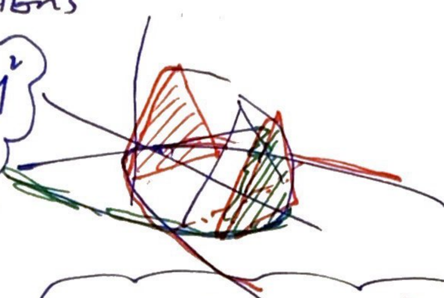
$x = ?$
1

- Intersections

$y^2 = 3 - 2y^2$

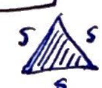
$3y^2 = 3$

$y = \pm 1$



- $\boxed{V = V_{\text{left half}} + V_{\text{right half}}}$

$\boxed{A_\Delta = \frac{\sqrt{3}}{4} s^2}$ area of an equilateral triangle



$$A_{left}(x) = \frac{\sqrt{3}}{4} s^2 \quad , \quad s = 2y$$

$$= \frac{\sqrt{3}}{4}(2y)^2 = \sqrt{3}\, y^2$$

$$\boxed{A(x) = \sqrt{3}\, x}$$

$s = 2y$

$$A_{right} = \sqrt{3}\, y^2$$

$$= \sqrt{3}\left(\frac{x-3}{-2}\right)$$

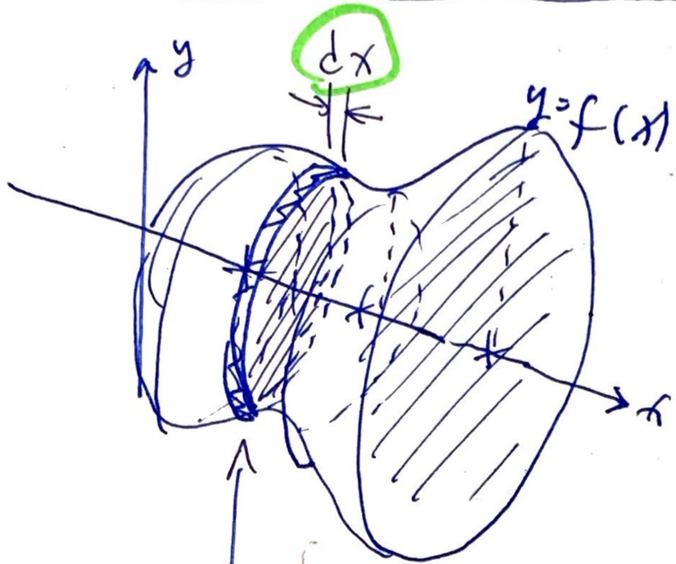
$$\boxed{A(x) = \frac{\sqrt{3}}{2}(3-x)}$$

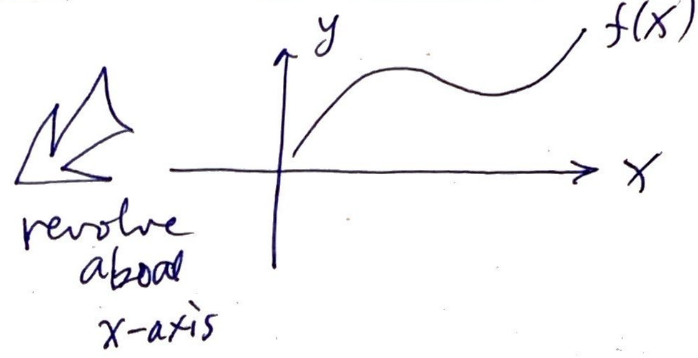
- Together

$$\boxed{V = \int_0^1 \sqrt{3}\, x \cdot dx + \int_1^3 \sqrt{3}\left(\frac{3-x}{2}\right) dx}$$

Set Up

II Volumes of Revolution – slices (disks)



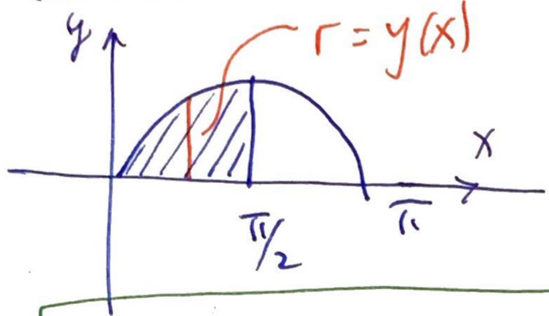


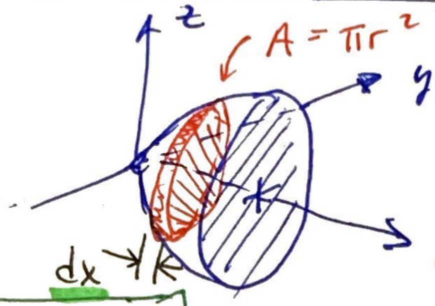
- Slice $\boxed{A(x) = \pi r^2}$ here $r = y$ so $A(x) = \pi \left[f(x)\right]^2$
- Volume $\Delta V(x) = A(x) \cdot \Delta x$

$\longrightarrow \lim_{\Delta x \to 0}$

$$\boxed{V = \pi \int_a^b \left[f(x)\right]^2 dx}$$

Ex let $y=f(x)$ where $f(x)=\sin(x)$. Now rotate $f(x)$ about the x-axis and find the volume from $x=0$ to $\pi/2$





$$\boxed{V=\pi\int_0^{\pi/2}\left[\sin(x)\right]^2 dx}$$

$$=\pi\int_0^{\pi/2}\sin^2(x)\,dx$$

use $\sin(x)=\frac{1}{2}\left(1-\cos(2x)\right)$

$$=\frac{\pi}{2}\int_0^{\pi/2}\left[1-\cos(2x)\right]dx$$

$$=\frac{\pi}{2}\,x\Big|_0^{\pi/2}-\frac{\pi}{2}\left(-\frac{\sin(2x)}{2}\right)\Big|_0^{\pi/2}$$

$$=\frac{\pi^2}{4}+\frac{\pi}{4}\Big(\underbrace{\sin\left(2\cdot\tfrac{\pi}{2}\right)}_{0}-\underbrace{\sin(0)}_{0}\Big)$$

$$V=\boxed{\frac{\pi^2}{4}}\ \text{cu. units}$$

$$\int\cos(2x)\,dx$$

$u=2x$

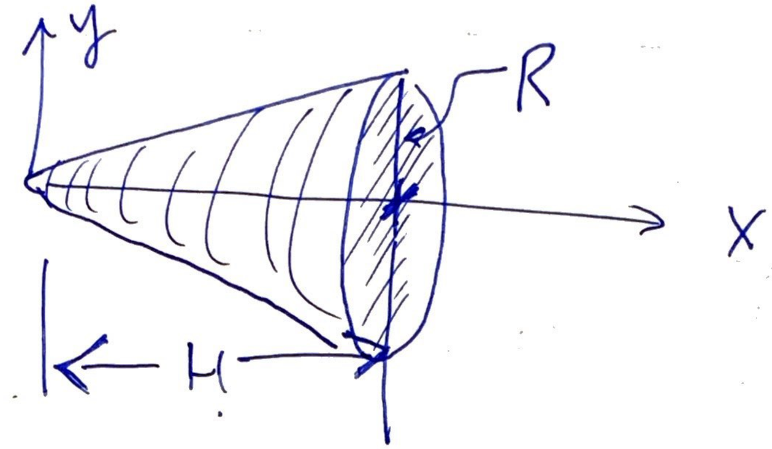
$du=2dx$

$$=\int\cos(u)\,\frac{du}{2}$$

$$=\frac{\sin(u)}{2}$$

Math 211 5a Name ______

2. Prove the equation of a cone's volume

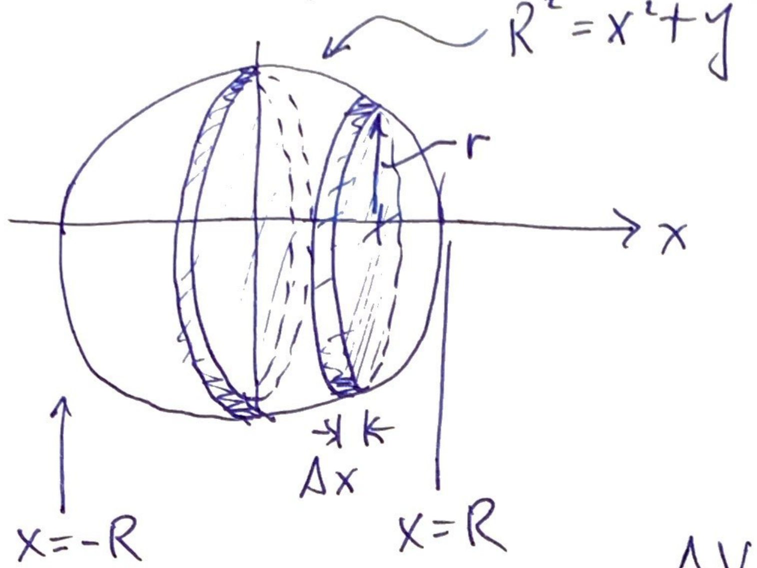


3. Rotate $y = x^2$ around the x-axis.
Find the volume this creates if we integrate from $x = 1$ to $x = 2$

$\boxed{\text{Ex}}$

Vol of a sphere

(i)



(ii)

$$\Delta V = A(x) \cdot \Delta x$$

$$\Delta V = \pi \left[r(x) \right]^2 \Delta x$$

but $r(x) = \sqrt{R^2 - x^2}$

since $r(x) = y(x)$

$$\Delta V = \pi \left(\sqrt{R^2 - x^2} \right)^2 \Delta x$$

or $\boxed{\Delta V = \pi (R^2 - x^2) \Delta x}$ slice

Total volume:

(iii)

$$V = \int_{x=-R}^{x=R} \pi (R^2 - x^2) dx$$

note symmetry

(iv)

$$= 2 \int_0^R \pi (R^2 - x^2) dx$$

$$= 2\pi R^2 x \Big|_0^R - 2\pi \frac{x^3}{3} \Big|_0^R$$

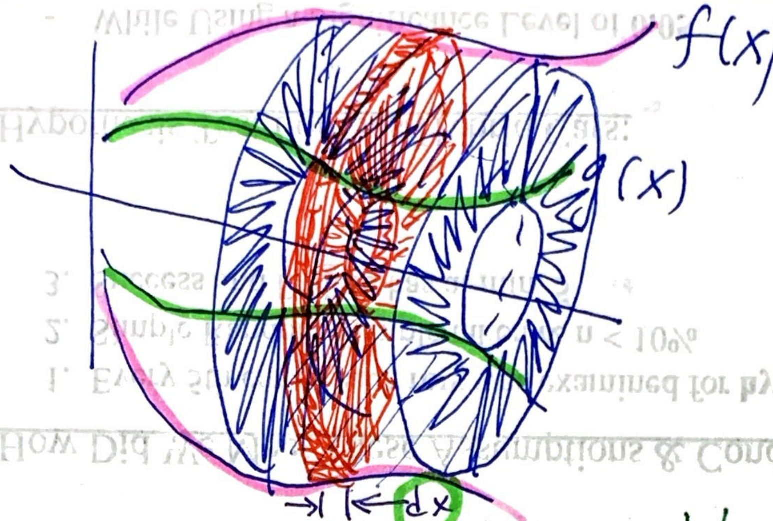
$$= 2\pi R^3 - \frac{2}{3} \pi R^3$$

$$= \frac{6}{3} \pi R^3 - \frac{2}{3} \pi R^3$$

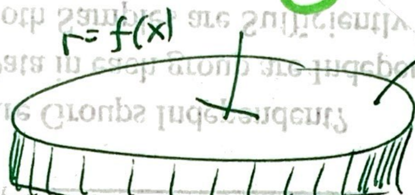
$$\boxed{V = \frac{4}{3} \pi R^3}$$

Congratulations! you've derived a formula you had no idea where it came from!!

III Volume of revolutions of areas between curves. "Washers"



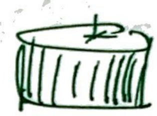
Jet Engine



$V = \pi f^2 \Delta$

$-$

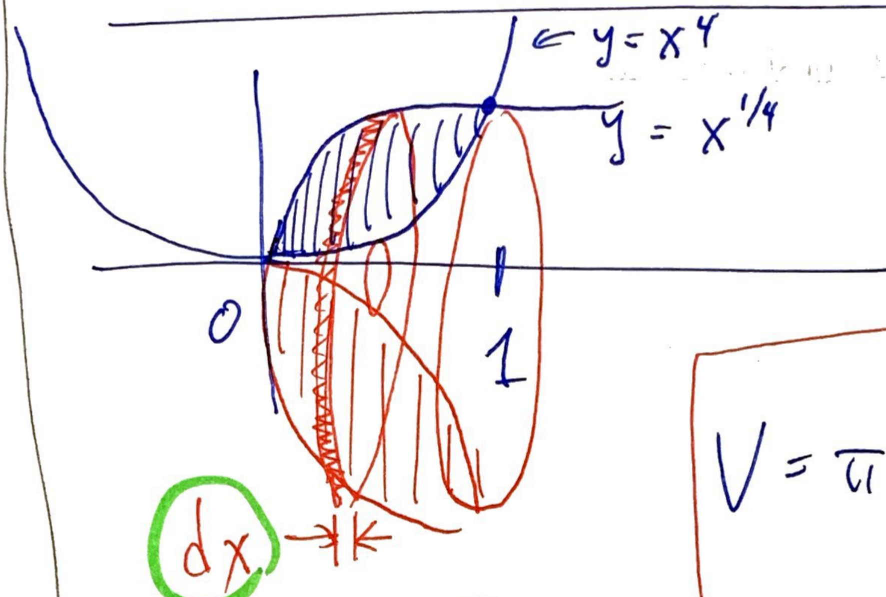
$V = \pi g^2$



r = g(x)

$$\boxed{V = \pi \int_a^b \left[f^2 - g^2 \right] dx}$$

Ex (Set-Up only) Find the volume generated by rotating the area between $y = x^{1/4}$ and $y = x^4$ from 0 to 1

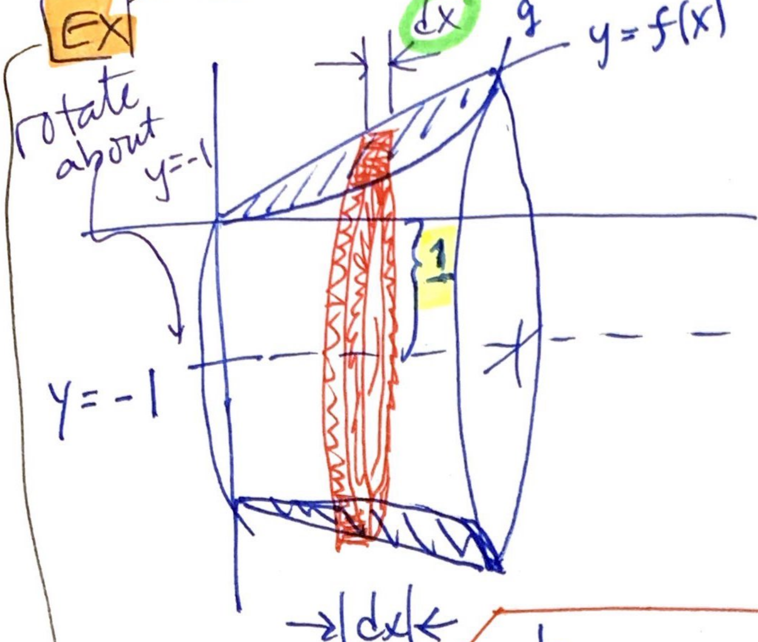


$$V = \pi \int_0^1 \left[(x^{1/4})^2 - (x^4)^2 \right] dx$$

Set-Up.

Rotation about a non-axis line.

EX



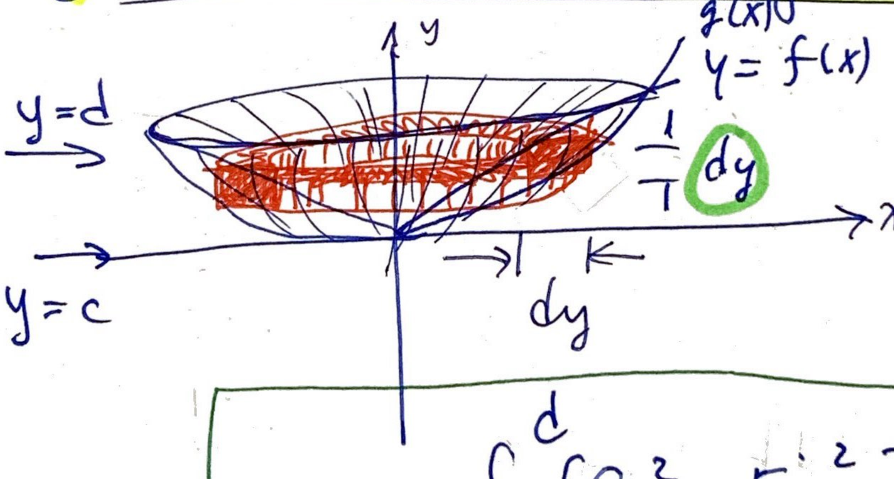
$$A_{outer} = \pi r_{out}^2$$

$$A_{outer} = \pi\,[1 + f(x)]^2$$

$$A_{inner} = \pi\,[1 + g(x)]^2$$

$$V = \pi \int_a^b \left[(1+f)^2 - (1+g)^2 \right] dx$$

Rotate about the y-axis



$$A_{outer} = \pi g^2$$

$$A_{inner} = \pi f^2$$

$$V = \pi \int_{y=c}^{d} \left[G^2 - F^2 \right] dy$$

where $F(y) = x$ is $y = f(x)$

where $G(y) = x$ is the solving of $y = g(x)$

must be functions of y also