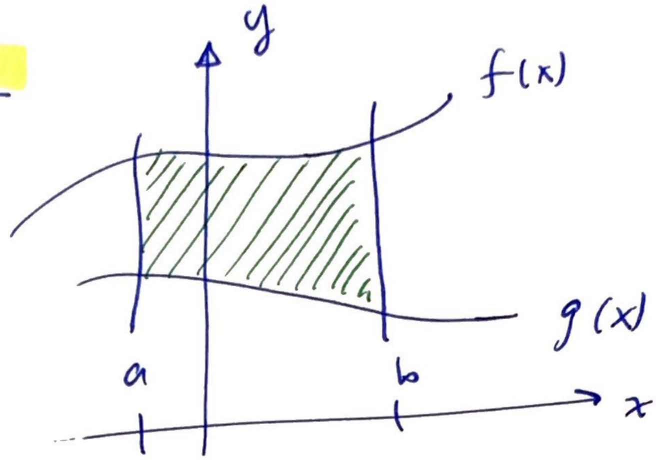
Chapter 5
Applications of Integration

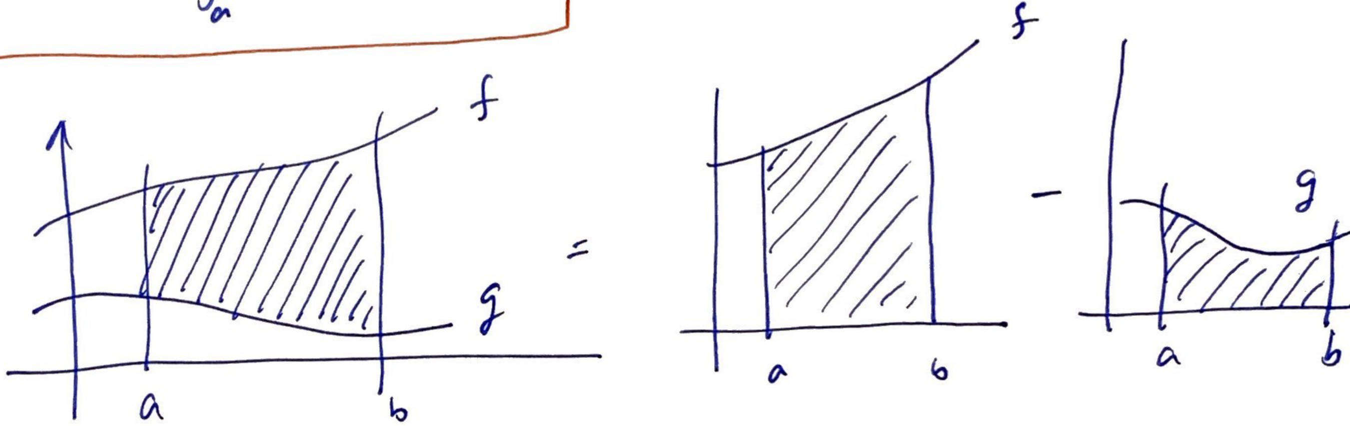
5.1 Area

I Area between curves

If between $x=a$ and $x=b$ we have $f(x) \geq g(x)$ then the area between the curves is

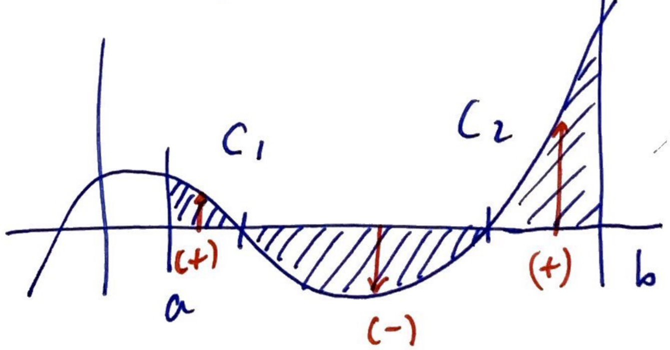
$$A = \int_a^b [f(x)-g(x)]\,dx$$





- OK if f or g goes negative since we are differencing

For

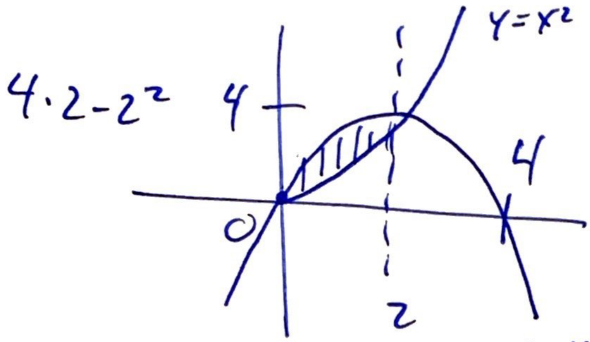


$$A_{TOT} = \int_a^{C_1} f\,dx + \int_{C_1}^{C_2} |f(x)|\,dx + \int_{C_2}^{b} f(x)\,dx$$

- In general $A = \int_a^b |f(x)|\,dx$, but there are details ... need C_1 & C_2

Math 211 CW 4d Name ______

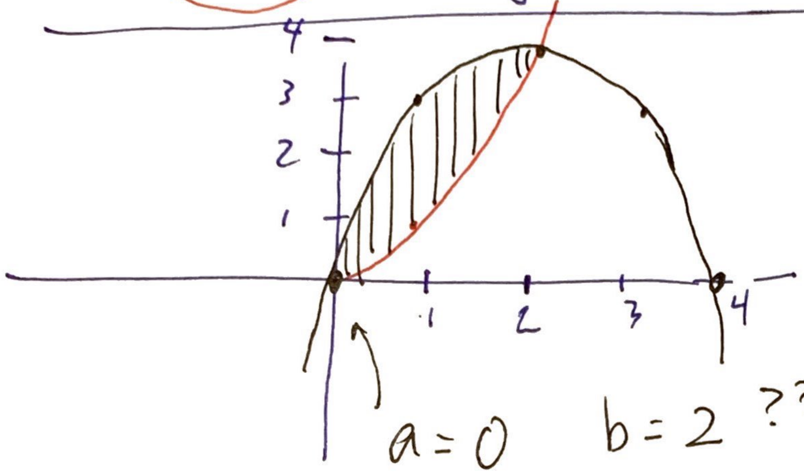
2 Find the area between $y = x^2$ & $y = 4x - x^2$



$$y = 4x - x^2$$
$$= x(4-x) \quad y = 0 \text{ @ } \underline{x = 0, 4}$$

$$\boxed{A = \int_{x=0}^{2} \left[(4x - x^2) - (x^2)\right] dx}$$

EX Sketch the region enclosed by $y = x^2$ and $y = 4x - x^2$ then find its area



x-int:

$0 = 4x - x^2$
$= x(4-x) \quad x = 0, 4$

$a = 0 \quad b = 2$??

intersection: $y = x^2$ set equal to $y = 4x - x^2$

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0 \rightarrow x = 0 \text{ or } 2$$

$$A = \int_{x=0}^{2} \left[(4x - x^2) - (x^2)\right] dx$$

$$= \int_0^2 [4x - 2x^2] dx$$

$$= 2\left\{\int_0^2 2x\, dx - \int_0^2 x^2 dx\right\}$$

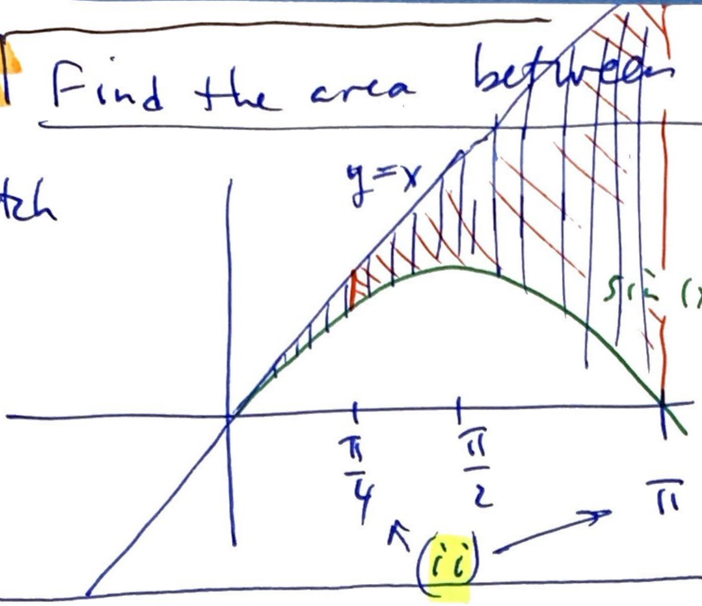
$$= 2\left\{\frac{\cancel{2}x^2}{\cancel{2}}\Big|_0^2 - \frac{x^3}{3}\Big|_0^2\right\}$$

$$= 2\left\{(2^2 - 0^2) - \left(\frac{2^3}{3} - \frac{0^3}{3}\right)\right\}$$

$$= 8 - \frac{16}{3} = \frac{24}{3} - \frac{16}{3} = \boxed{\frac{8}{3}}$$

Ex Find the area between "sin(x)" and "x" for $x \in [\frac{\pi}{4}, \pi]$

(i) Sketch



(iii) $$\boxed{A = \int_{\frac{\pi}{4}}^{\pi} [x - \sin(x)]\,dx}$$

(iv) $$= \int_{\frac{\pi}{4}}^{\pi} x\,dx - \int_{\pi/4}^{\pi} \sin(x)\,dx$$

$$= \frac{x^2}{2}\Big|_{\pi/4}^{\pi} - \Big[-\cos(x)\Big]_{\pi/4}^{\pi}$$

$$= \frac{1}{2}\left[\pi^2 - \left(\frac{\pi}{4}\right)^2\right] + \left[\cos(\pi) - \cos(\pi/4)\right]$$

-1

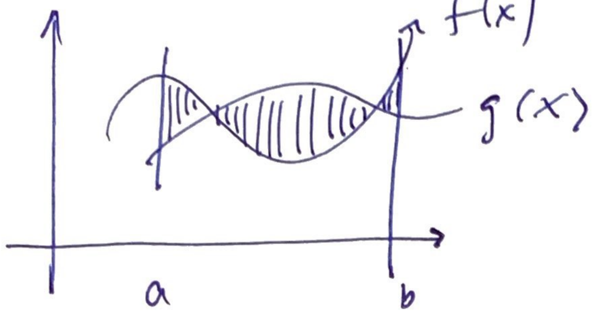
$$= \frac{\pi}{2}\left[1 - \frac{1}{16}\right] + \left[-1 - \sqrt{2}/2\right]$$

$\frac{16}{16} - \frac{1}{16}$

$$= \boxed{\frac{15\pi}{32} - 1 - \frac{\sqrt{2}}{2}} \approx \underline{\underline{2.92}} \text{ sq. units.}$$

II When f is not always $\geq g$

Q: what do we do if f and g intertwine?



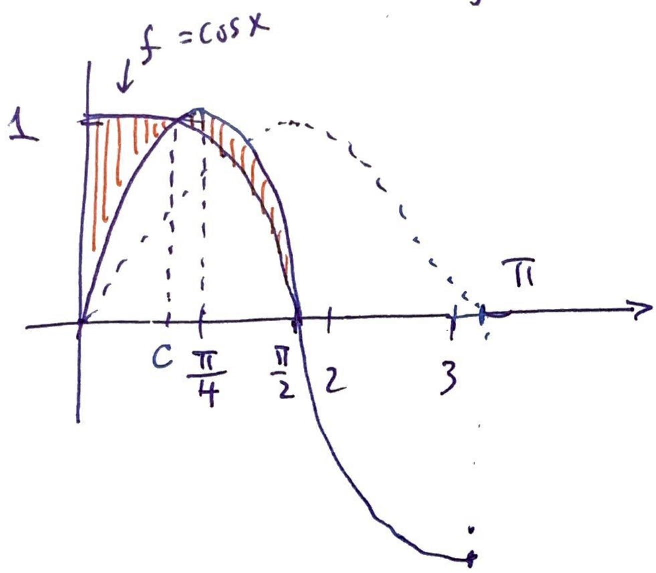
$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

(+) from now on

EX Sketch and calculate the area between

$y = \cos x$ and $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{2}$

f $\qquad$ g



$$A = \int_0^{\frac{\pi}{2}} \left| \cos x - \sin 2x \right| dx$$

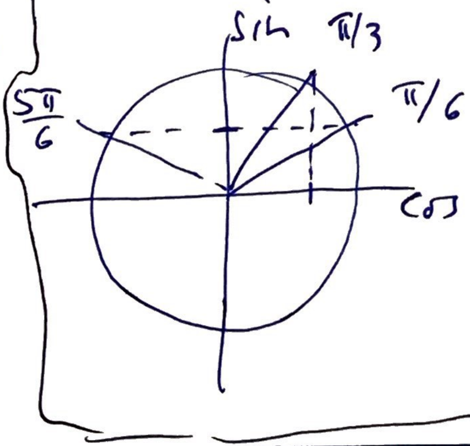
$$= \int_0^c (\cos x - \sin 2x)\, dx + \int_c^{\pi/2} (\sin 2x - \cos x)\, dx$$

• we need "c": $\cos x = \sin 2x$

$$\cos x = 2 \sin x \cos x$$

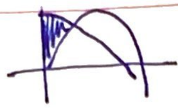
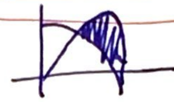
$$2 \sin x \cos x - \cos x = 0$$

$$\cos x \, (2 \sin x - 1) = 0$$



$$\begin{cases} \cos x = 0 & \boxed{x = \pi/2}, \ 3\pi/2 \ \cdots, \ -\frac{\pi}{2}, -\frac{3\pi}{2}, \cdots \\ 2 \sin x - 1 = 0 & \end{cases}$$

$\sin x = 1/2 \quad x = \frac{\pi}{2} \pm 2n\pi, \ \frac{5\pi}{6} \pm 2\pi \quad \boxed{x = \frac{\pi}{6}}$ c

 + 

So $\boxed{A = \int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx}$

• Evaluate

$$= \int_0^{\pi/6} \cos x\,dx - \int_0^{\pi/6} \sin 2x\,dx + \int_{\pi/6}^{\pi/2} \sin 2x\,dx - \int_{\pi/6}^{\pi/2} \cos x\,dx$$

$u = 2x$, $du = 2dx$ $\qquad$ $u = 2x$, $du = 2dx$

$$= \sin x \Big|_0^{\pi/6} - \int_{u=0}^{\pi/3} \sin u \left(\frac{du}{2}\right) + \int_{\pi/3}^{\pi} \sin u \left(\frac{du}{2}\right) - \sin x \Big|_{\pi/6}^{\pi/2}$$

$$= \sin x \Big|_0^{\pi 6} - \left(-\frac{\cos u}{2}\right)\Big|_0^{\pi/3} + \left(-\frac{\cos u}{2}\right)\Big|_{\pi/3}^{\pi} - \sin x \Big|_{\pi/6}^{\pi/2}$$

$$= \left(\sin\frac{\pi}{6} - \sin 0\right) + \left(\frac{\cos}{2}\pi/3 - \frac{\cos 0}{2}\right) + \left(\frac{-\cos\pi}{2} - \frac{(-\cos \pi/3)}{2}\right) - \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right)$$

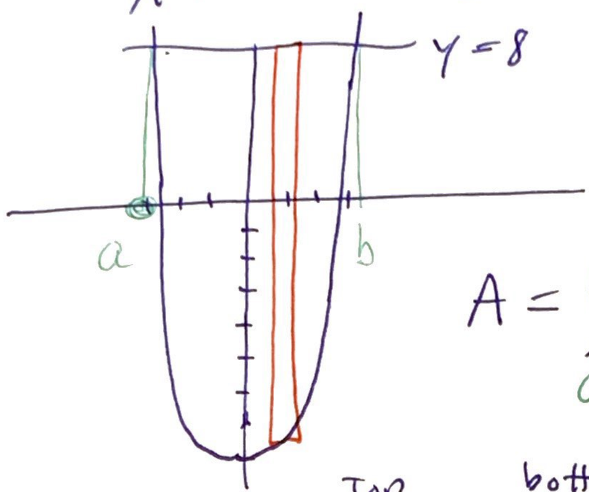
$$= \left(\frac{1}{2} - 0\right) + \left(\frac{1/2}{2} - \frac{1}{2}\right) + \left(+\frac{1}{2} + \frac{1/2}{2}\right) - \left(1 - 1/2\right)$$

$$= \cancel{\frac{1}{2}} + \frac{1}{4} - \cancel{\frac{1}{2}} + \frac{1}{2} + \frac{1}{4} - 1 + \frac{1}{2}$$

$$= \underbrace{\frac{1}{2} + \frac{1}{2} - 1}_{0} + \frac{1}{2} = \boxed{\frac{1}{2}}$$

EX

Find the area between $y = x^2 - 8$ and $y = 8$



$$A = \int_a^b \left[8 - (x^2 - 8)\right] dx$$

· intersection: $\underset{\text{Top}}{8} = \underset{\text{bottom}}{x^2 - 8} \Rightarrow 16 = x^2$ $\boxed{x = \pm 4}$

So $\boxed{A = \int_{-4}^{4} \left[8 - (x^2 - 8)\right] dx} = 2 \cdot \int_0^4 \left[8 - (x^2 - 8)\right] dx$

we have sym about y-axis.

$$A = 2\int_0^4 \left[16 - x^2\right] dx$$

$$= 2\left\{ 16x\Big|_0^4 - \frac{x^3}{3}\Big|_0^4 \right\}$$

$$= 2\left\{ (16 \cdot 4 - 16 \cdot 0) - \left(\frac{4^3}{3} - \frac{0^3}{3} \right) \right\}$$

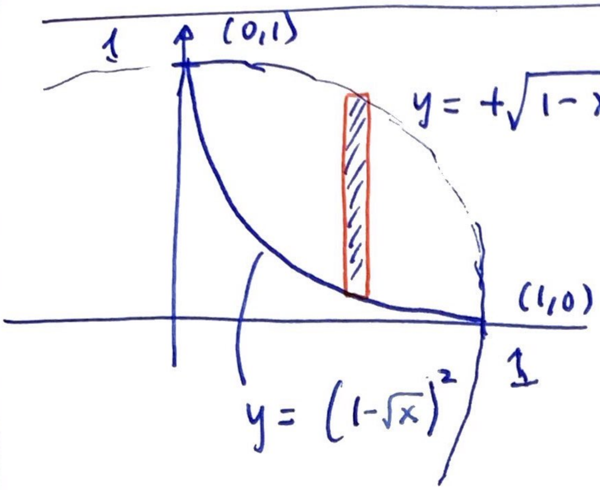
$$= 2\left\{ (64 - 0) - \left(\frac{64}{3} - 0 \right) \right\}$$

$$= 2\left\{ 64\left(1 - \frac{1}{3}\right) \right\}$$

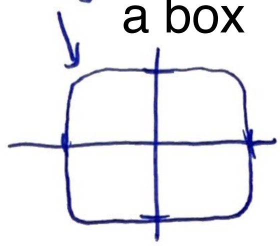
$$= 2 \cdot 64 \left(\frac{2}{3}\right)$$

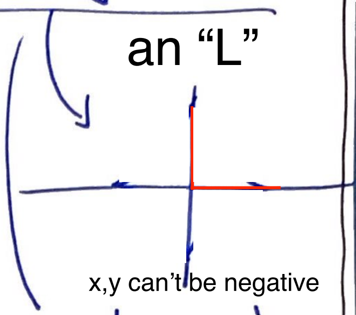
$$= \frac{240 + 16}{3} = \boxed{\frac{256}{3}}$$

Ex Find the area, in the first Quadrant, between the curves $x^2+y^2=1$ & $x^{1/2}+y^{1/2}=1$



If ↓ $x^{100}+y^{100}=1$

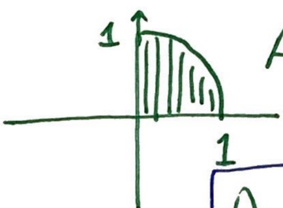




$x^{\frac{1}{100}}+y^{\frac{1}{100}}=1$

$$\boxed{A=\int_0^1\left[\sqrt{1-x^2}-(1-\sqrt{x})^2\right]dx}$$

- Need to wait for Chpt 7 (Calc II)
- But we still can find the result.



$A=\pi r^2\Big|_{r=1}$

$$\boxed{A_{I}=\frac{\pi}{4}}$$

$(1-2\sqrt{x}+\sqrt{x}^2)$

$=(1-2\sqrt{x}+x)$

$\int_0^1(1-2\sqrt{x}+x)\,dx$

$=\int_0^1 1\cdot dx-2\int_0^1\sqrt{x}\,dx+\int_0^1 x\,dx$

$=x\Big|_0^1-2\frac{x^{3/2}}{3/2}\Big|_0^1+\frac{x^2}{2}\Big|_0^1$

$=(1-0)-\frac{2\cdot 2}{3}(1^{3/2}-0)+\left(\frac{1^2}{2}-0\right)$

$=1-\frac{4}{3}+\frac{1}{2}$

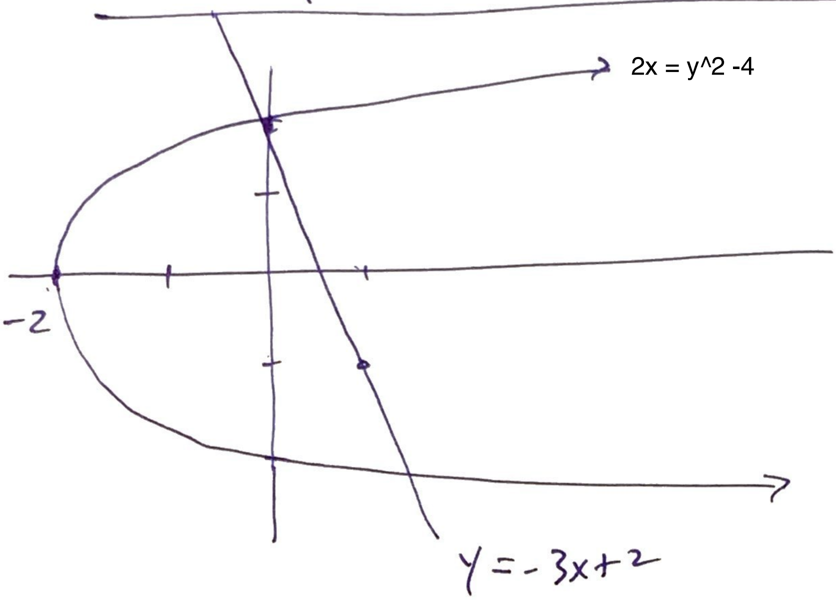
$=\frac{6}{6}-\frac{8}{6}+\frac{3}{6}=\boxed{\frac{1}{6}}$

$$\boxed{A=\frac{\pi}{4}-\frac{1}{6}}$$

④ integration with respect to y :

EX Calculate the area between $2x = y^2 - 4$ and $y = -3x + 2$

sideways



$x = \frac{1}{2} \cdot y^2 - 2$

• x-int @ y = 0 :

$x = \frac{1}{2} \cdot 0^2 - 2$

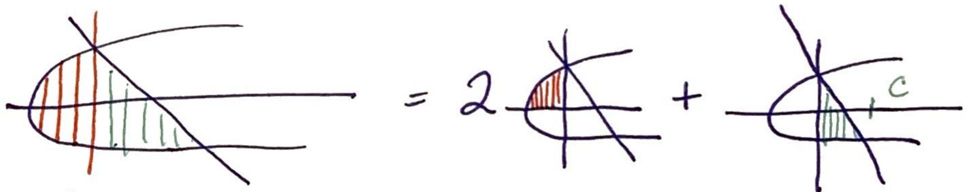
$x = -2$

• y-int : @ x = 0

$2 \cdot 0 = y^2 - 4$

$y = \pm 2$

"Traditional"



$\int \left(+\sqrt{2x+4} - (-\sqrt{2x+4})\right) dx$

$\sqrt{2x+4} + \sqrt{2x+4}$

$\begin{cases} y = \pm\sqrt{2x+4} & \text{top} \\ y = -\sqrt{2x+4} & \text{bot} \end{cases}$

Top: $y = \sqrt{2x+4}$
Bot: $y = 0$

Top: $y = -3x+2$
Bot: $-\sqrt{2x+4}$

$$\boxed{A = 2 \cdot \int_{x=-2}^{x=0} \sqrt{2x+4}\, dx + \int_{x=0}^{x=c=\frac{14}{9}} \left[(-3x+2) - (-\sqrt{2x+4})\right] dx}$$

• find c : substitute $y = -3x+2$ into $2x = y^2 - 4$

$\Rightarrow \quad 2x = (-3x+2)^2 - 4$

$2x = 9x^2 - 12x + 4 - 4$

$9x^2 - 14x = 0 \Rightarrow x(9x - 14) = 0$ x - coord $\boxed{x = +14/9}$

Now try side wayze strips …

- Intersection: $x_{left} = x_{right}$

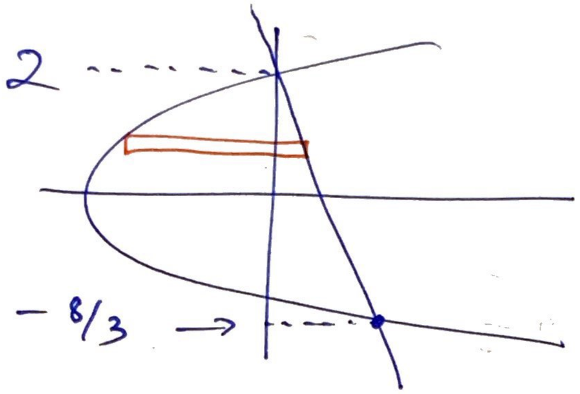
$$\frac{y^2-4}{2} = \frac{y-2}{-3}$$

Solve $-3y^2+12 = 2y-4 \rightarrow -3y^2-2y+16=0$

$$\boxed{3y^2+2y-16=0}$$

$$\begin{array}{c|ccc} 2 & 3 & 2 & -16 \\ & & 6 & 16 \\ \hline & 3 & 8 & 0 \end{array}$$

$3y+8=0$ y-coord $\boxed{y = -\frac{8}{3}}$



- Back to the area integral:

$$\boxed{A = \int_{y=-\frac{8}{3}}^{2} \left[\left(\frac{y-2}{-3}\right) - \left(\frac{y^2-4}{2}\right) \right] dy}$$

simpler that vertical strips

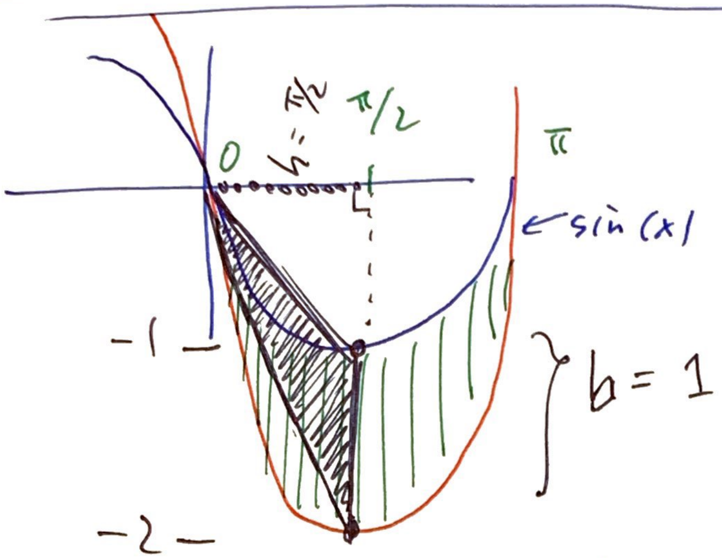
$$= \int_{-8/3}^{2} \left[-\frac{y^2}{2} + 2 - \frac{y}{3} + \frac{2}{3} \right] dy \ * \frac{-6}{-6}$$

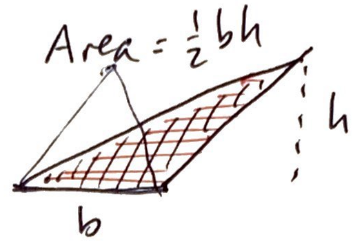
$$= \frac{1}{-6} \int_{-8/3}^{2} \left[3y^2 - 12 + 2y - 4 \right] dy$$

$$A = -\frac{1}{6} \int_{-8/3}^{2} \left[3y^2+2y-16 \right] dy = \boxed{\frac{154}{81}}$$

⊛ approximation via geometry

Ex Find the area between $-\sin(x)$ and $-2\sin(x)$





(a) Estimate $A \approx 2A_{\triangle} = 2\left(\frac{1}{2}(1)\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{2} \boxed{\approx 1.5}$

(b) $A = \int_0^{\pi} \left[(-\sin(x)) - (-2\sin(x))\right] dx$

$= \cos(x)\Big|_0^{\pi} + 2(-\cos(x))\Big|_0^{\pi}$

$= \cos(\pi) - \cos(0) - 2\left[\cos(\pi) - \cos(0)\right]$

$= -1 - 1 - 2\left[-1 - 1\right]$

$= -2 - 2[-2]$

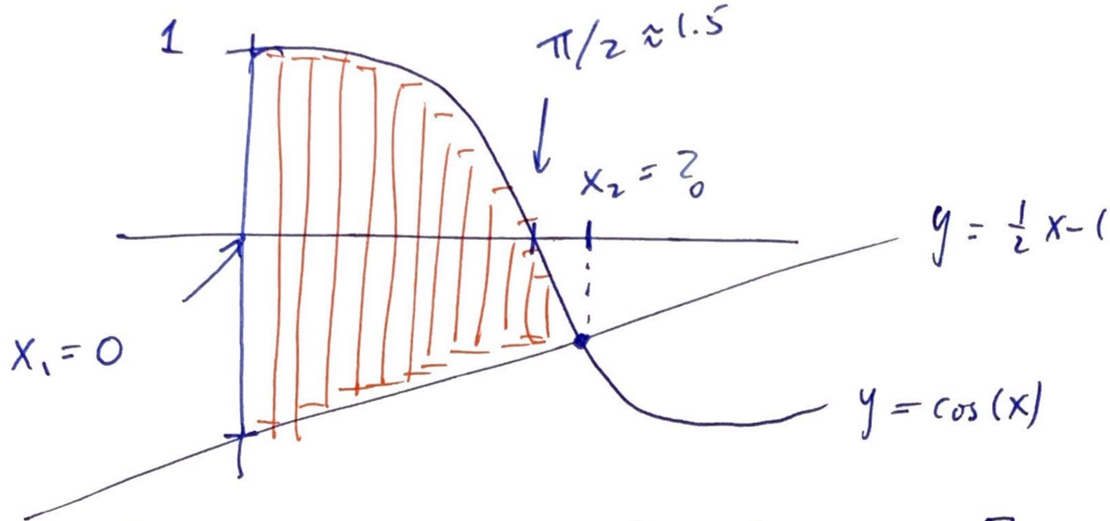
$= -2 + 4$

$= \boxed{2}$ vs. $\boxed{1.5}$

^ Crude but not bad for a ball park area

⊛ Estimating Intersections

Ex Set up the area integral for the area between $y = \cos(x)$ and $y = \frac{1}{2}x - 1$



• To find x_2: $\underbrace{\cos(x)}_{y_{upper}} = \underbrace{\frac{1}{2}x - 1}_{y_{lower}}$ } difficult to solve analytically

• Approximate

Hunt and Peck

$\cos(1.5)$	vs.	$\frac{1}{2}(1.5) - 1$
0.0707		−0.25
$\cos(1.9)$	vs	$\frac{1}{2}(1.9) - 1$
−0.323		−0.05
$\cos(2.2)$	vs	$\frac{1}{2}(2.2) - 1$
−0.588		+0.1
$\cos(1.7)$	vs.	$\frac{1}{2}(1.7) - 1$
−0.1288		−0.15
$\cos(1.75)$	vs.	$\frac{1}{2}(1.75) - 1$
−0.128		−0.125

go back

$x_2 \approx 1.75$

Newton's Method

let $f(x) = \cos(x) - \frac{x}{2} + 1$

$f'(x) = -\sin(x) - \frac{1}{2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = 1.5 - \frac{\cos(1.5) - \frac{1.5}{2} + 1}{-\sin(1.5) - \frac{1}{2}} = 1.714$$

$$x_3 = 1.714 - \frac{\cos(1.714) - \frac{1.714}{2} + 1}{-\sin(1.714) - 1/2}$$

$$= 1.714 - \frac{0.000285}{-1.48976}$$

$$= 1.71419$$

Ex. cont'd

$$A = \boxed{\int_0^{1.714} \left[\cos(x) - \left(\frac{x}{2} - 1\right)\right] dx}$$

The set up.