4.5 The Substitution Rule {chain rule in "Reverse"}

We know that $\int \cos(x)\,dx = \sin(x) + C$. But what is $\int \sin(\sqrt{2}x)\,dx = ?$

I This is where we introduce the "u-sub" rule

Ex $\int \sin(\sqrt{2}x)\,dx$

let $u = \sqrt{2}x$ then $du = \sqrt{2}\,dx$ or $dx = \frac{du}{\sqrt{2}}$

Now the integral becomes

$$\int \sin(u)\left(\frac{du}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\int \sin(u)\,du$$

$$= \frac{1}{\sqrt{2}}(\cos(u) + c)$$

unsubstitute

$$= \frac{1}{\sqrt{2}}(\cos(\sqrt{2}x) + c)$$

$$\int \sin(\sqrt{2}x)\,dx = \boxed{\frac{1}{\sqrt{2}}\cos(\sqrt{2}x) + c}$$

Ex evaluate $\int 3x^2\sqrt{1+x^3}\,dx$

let $u = 1+x^3$ then $du = 3x^2 dx$

so the integral becomes

$$\int \sqrt{u}\,\underbrace{(3x^2 dx)}$$

$$= \int \sqrt{u}\;du$$

$$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \qquad \tfrac{1}{2}+1 = \tfrac{3}{2}$$

$$= \boxed{\frac{2\left(\sqrt{1+x^3}\right)^3}{3} + C}$$

Ex Evaluate $\int x^3\sqrt{x^2+1}\,dx$

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let $u = x^2+1 \rightarrow du = 2x\,dx \rightarrow dx = \dfrac{du}{2x}$

$$\int x^3\sqrt{u}\left(\frac{du}{2x}\right)$$

$$= \frac{1}{2}\int x^2\sqrt{u}\,du$$

but $x^2 = u-1$

$$= \frac{1}{2}\int (u-1)\sqrt{u}\,du$$

$$= \frac{1}{2}\int u\sqrt{u}\,du - \frac{1}{2}\int\sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{3/2}\,du - \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\frac{u^{3/2+1}}{\frac{3}{2}+1} - \frac{1}{2}\frac{u^{1/2+1}}{\frac{1}{2}+1} + C$$

$$\frac{a}{\frac{b}{c}} = \frac{ac}{b}$$

$$= \frac{1}{2}\frac{u^{5/2}}{5/2} - \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{2}\frac{2}{5}u^{5/2} - \frac{1}{2}\frac{2}{3}u^{3/2} + C$$

u-sub

$$= \boxed{\frac{1}{5}\left(\sqrt{x^2+1}\right)^5 - \frac{1}{3}\left(\sqrt{x^2+1}\right)^3 + C}$$

Theory

Thm If $\boxed{u = g(x)}$ is diff'ble on range I
and if $f(x)$ is continuous on I also
then $\boxed{\int f(g(x))\, g'(x)\,dx = \int f(u)\,du}$

proof:

I. $\boxed{1^{st} \text{ examine } I = \int f(g(x))\, g'(x)\,dx}$

- Note $\frac{d}{dx}\left[F(g(x))\right] = F'(g)\cdot g'$ chain Rule
- now let $F' = f$

$\boxed{\frac{d}{dx}\int f = f}$

so then $I = \int \overbrace{f}^{F'}(g(x))\, \underbrace{g'(x)\,dx}_{dg}$

becomes $= \int F'(g)\,dg \Leftarrow \boxed{\frac{dg(x)}{\cancel{dx}}\cancel{dx} = dg}$

$= F(g) + C$ by FTC·I

Now let $\boxed{u = g(x)} \rightarrow du = g'(x)\,dx$

then $\int F'(g(x))\, g'(x)\,dx$

$= \int F'(u)\,du$

but $F' = f$

LHS RHS

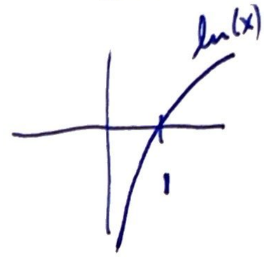
$$\boxed{\int f(g(x))\, g'(x)\,dx = \int f(u)\,du}$$ Q.E.D.

5

In Chapter 6 we reveal that

$$\boxed{\frac{d\ln(x)}{dx} = \frac{1}{x}}$$

$$\begin{cases} \ln(x) = \log_e(x) \\ \log(x) = \log_{10}(x) \end{cases}$$



So In reverse (antiderivation)

$$\boxed{\int \frac{1}{x}\,dx = \ln(x) + c}$$

Ex Find $\int \tan(x)\,dx$

$\int \frac{\sin(x)\,dx}{\cos(x)}$, now let $u = \cos(x)$ so $du = -\sin(x)\,dx$

$$= \int \frac{-du}{u}$$

$$= -\ln(u) + c$$

$$= -\ln(\cos(x)) + c$$

$$= \ln(\cos(x))^{-1} + c$$

$$= \ln\left(\frac{1}{\cos(x)}\right) + c$$

$$= \ln(\sec(x)) + c$$

$$\boxed{b \cdot \log(a) = \log(a^b)}$$

$$\begin{cases} \text{let } b = -1 \\ \log(a) = \log(a^{-1}) \\ \quad = \log\left(\frac{1}{a}\right) \end{cases}$$

So $\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + c}$

II Definite Integrals

For definite Integrals we have two choices

A. Evalute the integral, w/o limits, via a u-sub, then unsubstitute and reapply the limits.

B. Change the limits when we do the u-sub.

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(b)}^{u=g(a)} f(u)\,du$$

Ex of method A.

$\boxed{1+t^2 = \sec^2}$

$$\int_0^{\pi/3} \frac{\sin(z) + \sin(z)\tan^2(z)}{\sec^2(z)}\,dz$$

$$= \int_0^{\pi/3} \frac{\sin(z)\overbrace{(1+\tan(z))}^{\sec^2(z)}}{\sec^2(z)}\,dz$$

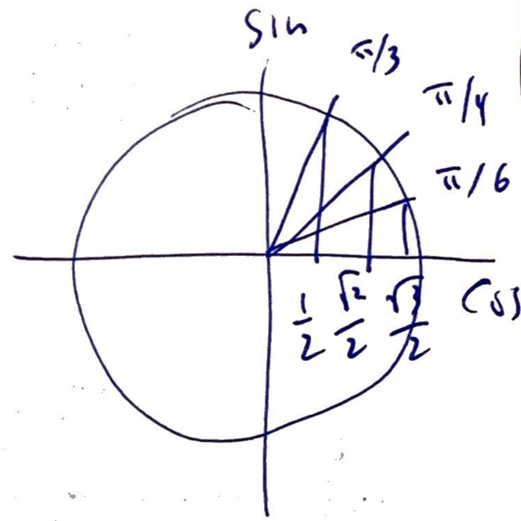
$$= \int_0^{\pi/3} \sin(z)\,dz$$

$$= -\cos(z)\Big|_0^{\pi/3}$$

$$= -\cos\left(\tfrac{\pi}{3}\right) - \left(-\cos(0)\right)$$

$$= -\tfrac{1}{2} - (-1)$$

$$= \boxed{\tfrac{1}{2}}$$



Hmm No u-sub needed... after all...

Ex Evaluate $\displaystyle\int_{x=1}^{4} 3x^2\sqrt{1+x^3}\,dx$

Method **A**

$u = 1+x^3 \longrightarrow du = 3x^2\,dx$ {previous example}

$$\int_{x=1}^{x=4} \sqrt{u}\,du$$

integrate

$$= \left.\frac{u^{3/2}}{3/2}\right|_{x=1}^{x=4}$$

un-substitute

$$= \boxed{\left.\frac{2\left(\sqrt{1+x^3}\right)^3}{3}\right|_{x=1}^{x=4}}$$

evaluate limits...

$$= \frac{2}{3}\left[\left(\sqrt{1+4^3}\right)^3 - \left(\sqrt{1+1^3}\right)^3\right]$$

$$= \frac{2}{3}\left[\sqrt{65}^{\,3} - \sqrt{2}^{\,3}\right]$$

$(\sqrt{a})^3 = \sqrt{a}\sqrt{a}\sqrt{a}$
$= a\sqrt{a}$

$$= \boxed{\frac{2}{3}65\sqrt{65} - \frac{2}{3}2\sqrt{2}}$$

$$= \boxed{\frac{130\sqrt{65}}{3} - \frac{4\sqrt{2}}{3}}$$

Ex cont.

method B

$$\int_{x=1}^{x=4} 3x^2 \sqrt{1+x^3}\, dx$$

let $u = 1 + x^3$, $du = 3x^2 dx$

$$\int_{x=1}^{x=4} \sqrt{u}\, du$$

$$= \int_{u=1+(1)^3}^{u=1+(4)^3} \sqrt{u}\, du$$

$$= \boxed{\int_{u=2}^{u=65} u^{1/2}\, du}$$

Totally converted to "u"

No more "x"

$$= \int_{2}^{65} u^{1/2}\, du$$

$$= \left.\frac{u^{3/2}}{3/2}\right|_{2}^{65}$$

$$= \frac{2}{3}\left[65^{3/2} - 2^{3/2}\right]$$

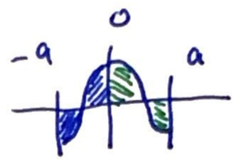
$$= \frac{2 \cdot 65\sqrt{65}}{3} - \frac{2 \cdot 2\sqrt{2}}{3}$$

III Symmetry and Integrals

If $f(x)$ is continuous on a symmetric interval $[-a, a]$ then

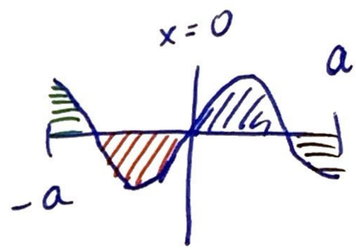
(a) If $f(x)$ = even function

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx$$



(b) If $f(x)$ = odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$



proof for (b): Recall $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx$

Then $\int_{-a}^{a} f\,dx = \int_{-a}^{0} f\,dx + \int_{0}^{a} f\,dx$

$$= -\int_{x=0}^{x=-a} f(x)\,dx + \int_{0}^{a} f\,dx$$

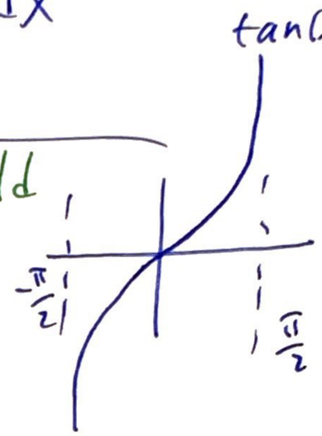
let $u = -x$
$du = -dx$

$$= -\int_{u=0}^{u=a} \underbrace{f(-u)}_{\text{odd}}(-du) + \int_{0}^{a} f(x)\,dx$$

$$= \underbrace{\int_{u=0}^{a} -f(u)\,du}_{-I} + \underbrace{\int_{x=0}^{a} f(x)\,dx}_{I} = -I + I = \boxed{0}$$

Ex Evaluate

$$\int_{-\pi/4}^{\pi/4} \left(x^3 + x^4 \tan(x)\right) dx$$



$$= \int_{-\pi/4}^{\pi/4} \underbrace{x^3}_{\text{odd}} dx + \int_{-\pi/4}^{\pi/4} \underbrace{\overset{\text{even}}{x^4} \overset{\text{odd}}{\tan(x)}}_{\text{odd}} dx$$

Sym interval

$$= 0 + 0$$

even · odd = odd

odd · odd = even

Ex

$$\int_0^1 x\sqrt{1+x^2}\, dx$$

Just another u-sub example

$u = 1 + x^2$, $du = 2x\,dx$

$$= \int_{u=1+0^2}^{u=1+1^2} \sqrt{u} \left(\frac{du}{2}\right)$$

$$= \boxed{\frac{1}{2}\int_1^2 \sqrt{u}\, du}$$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} \Bigg|_1^2$$

$$= \frac{1}{3}\left[\sqrt{2}^{\,3} - \sqrt{1}^{\,3}\right]$$

$$= \boxed{\frac{2\sqrt{2}}{3} - \frac{1}{3}}$$