4.3 The Fundamental Theorem of Calculus

Thm: "The Fundamental Theorem of Integral Calculus I" Pt.

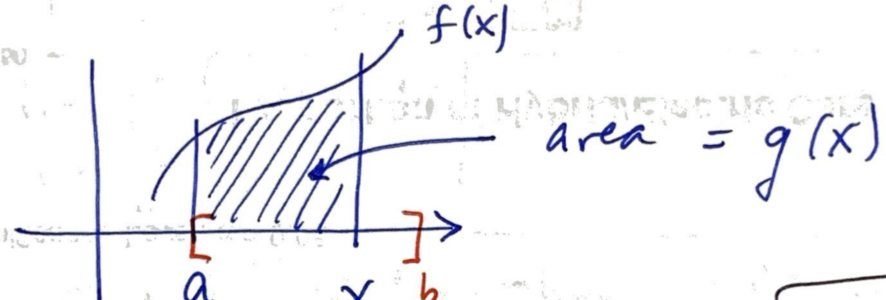
If $f(x)$ is a continuous function on $[a,b]$

then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$

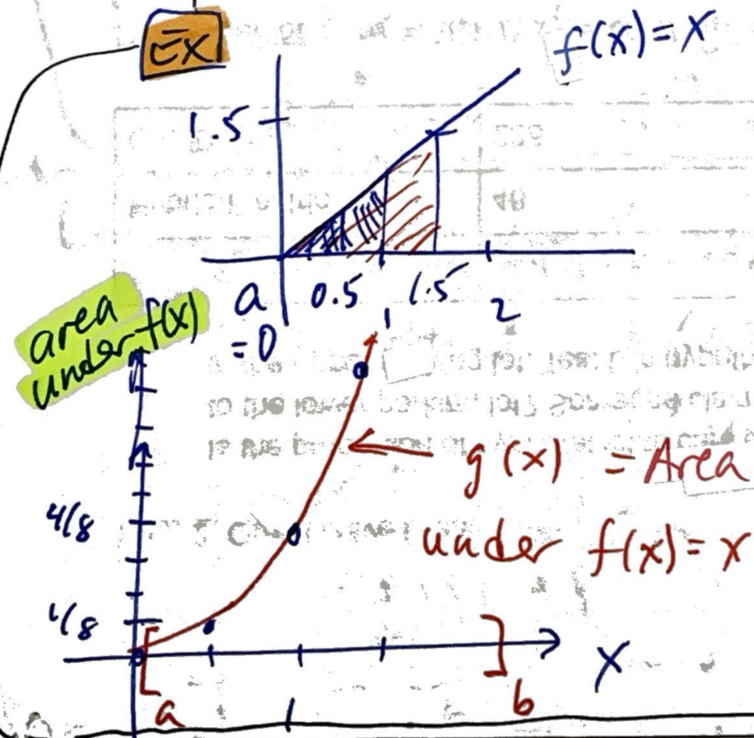
Integration variable

and is differentiable on (a,b)

Furthermore $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$



Ex



x	$y=x$ $f(x)=x$	area $g(x)$
0	0	0
0.5	0.5	$\frac{1}{2}(0.5)(0.5) = 0.125$
1.0	1.0	$\frac{1}{2}(1)(1) = 0.5$
1.5	1.5	$\frac{1}{2}(1.5)(1.5) = 9/8 = 1.125$
2.0	2.0	$\frac{1}{2}(2.0)(2.0) = 2.0$
⋮	⋮	⋮

Proof: We need $\dfrac{d\,g(x)}{dx}$ where $g(x) = \int_a^x f(t)\,dt$

- by Def. then

$$= \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)$$

becomes

$$= \lim_{h \to 0} \left(\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \right)$$

- split 1st integral up

$$= \lim_{h \to 0} \frac{\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt - \cancel{\int_a^x f(t)\,dt}}{h}$$

So

$$= \lim_{h \to 0} \left[\frac{1}{h} \int_x^{x+h} f(t)\,dt \right]$$

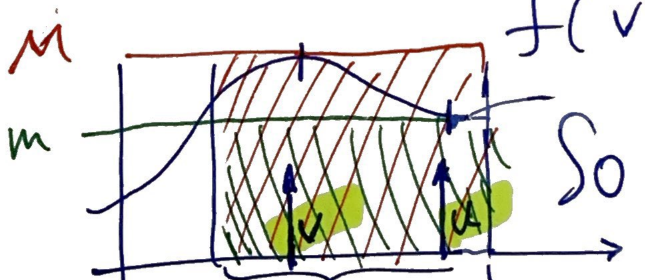
- The Extreme Value thm said that if $f(x)$ is continuous on $[a,b]$ then it attains its max M and min m on $[a,b]$

- ~~Introduce u and v~~

Let M and m be the max and min of $f(x)$

such that $f(u) = \min = m$

$f(v) = \max = M$



So $\boxed{m \cdot h} \leq \int_x^{x+h} f(t)\,dt \leq \boxed{M \cdot h}$

showes upper & lower bounds...

Proof (cont.)

- Next Substitute for $m = f(u)$ & $M = f(v)$

$$\Longrightarrow f(u)\cdot h \le \int_x^{x+h} f(t)\,dt \le f(v)\cdot h$$

$$\therefore \div h \qquad \boxed{f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)}$$

- But Recall $\int_x^{x+h} f(t)\,dt$ is just $g(x+h) - g(x)$

$$\Longrightarrow \boxed{f(u) \le \frac{1}{h}\left[g(x+h) - g(x)\right] \le f(v)}$$

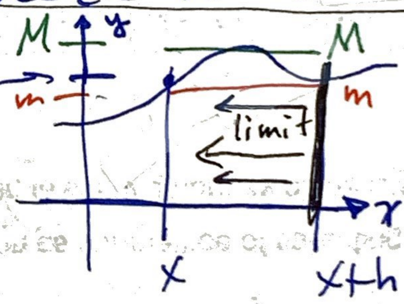
Now in the limit as $h \to 0$ we get $\frac{dg}{dx}$

$$\boxed{f(x) \le \frac{dg(x)}{dx} \le f(x)}$$

- But as we squeeze $h \to 0$ we have $u \to x$
likewise as we squeeze $h \to 0$ we have $v \to x$ also

as "x+h" slides over the M and m are changing

$f(x)$



Slide the $\underline{x+h}$ bar towards x bar

- By the squeeze thm if $f \le \frac{dg}{dx} \le f$ then

$$\boxed{\frac{dg(x)}{dx} = f(x)}$$

So $\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$ Q.E.D

Ex diff't $\int_1^x \frac{1}{t^3+1}\,dt$

F.T.C.I $\frac{d}{dx}\left(\int_1^x \frac{1}{t^3+1}\,dt\right) = \frac{1}{x^3+1}$ ✓

Ex Find $\frac{d}{dx}\left(\int_1^{x^4} \cos^2(\theta)\,d\theta\right)$

let $u = x^4$ then $du = 4x^3\,dx$

So $\frac{d}{dx} = \frac{d}{\left(\frac{du}{4x^3}\right)} = 4x^3 \frac{d}{du}$

looks like the chain rule

$$\frac{d}{dx}\int_1^{x^4} \cos^2(\theta)\,d\theta = \cos^2(x^4)\cdot\frac{dx^4}{dx}$$

chain rule

FTC·I with "chain Rule"

$$\boxed{\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(x)\cdot\frac{dh(x)}{dx}}$$

Thm **F. Thm of C part II**

If $f(x)$ is continuous on $[a,b]$

then $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F(x)$ is the antiderivative of $f(x)$

i.e., $\frac{dF(x)}{dx} = f(x)$

Ex Evaluate $\int_0^4 (4-t)\sqrt{t}\,dt$ by finding an anti-derivative

Find F such that $\frac{dF}{dx} = (4-x)\sqrt{x}$.

- Distribute $\frac{dF}{dx} = 4x^{1/2} - x^{3/2}$

  - let F_1 be such that $\frac{dF_1}{dx} = 4x^{1/2}$

    Try $F_1 = Ax^n$, then $\frac{dF_1}{dx} = Anx^{n-1}$

    → match $An = 4$ and $n-1 = 1/2$

    → solve for n 1st : $n = \frac{1}{2} + 1 = 3/2$

    → solve for A next: $A \cdot \frac{3}{2} = 4 \implies A = \frac{8}{3}$

    → So $\boxed{F_1 = \frac{8}{3}x^{3/2}}$

  - let F_2 be such $F_2' = -x^{3/2}$; Let $F_2 = Bx^m$, then $\frac{dF_2}{dx} = mBx^{m-1}$

    → match $Bm = -1$ and $m-1 = 3/2$

    → $m = \frac{3}{2} + 1 = \frac{5}{2}$ and so $B \cdot \frac{5}{2} = -1 \implies B = \frac{-2}{5}$

    thus $\boxed{F_2 = -\frac{2}{5}x^{5/2}}$

Together $\boxed{F = \frac{8}{3}x^{3/2} - \frac{2}{5}x^{5/2}}$

Armed with the anti-derivative proceed to solve

Now that we have an anti-derivative we apply FTC pt II

$$\int_0^4 \underbrace{(4-t)\sqrt{t}}_{f(x)}\,dt = F(4) - F(0)$$

where $\boxed{F(x) = \frac{8}{3}x^{3/2} - \frac{2}{5}x^{5/2}}$

So

$$\int_0^4 (4-t)\sqrt{t}\,dt = \left(\frac{8}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\Bigg|_{x=4} - \left(\frac{8}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\Bigg|_{x=0}$$

$$= \underbrace{\left[\frac{8}{3}\sqrt{4}^3 - \frac{2}{5}(\sqrt{4})^5\right]}_{F(b)} - \underbrace{\left[\frac{8}{3}(\sqrt{0})^3 - \frac{2}{5}(\sqrt{0})^5\right]}_{F(a)}$$

$$= \frac{64}{3} - \frac{2}{5}2^5 - 0$$

$$= \frac{64}{3} - \frac{2^6}{5}$$

$$= \frac{64}{3} - \frac{64}{5}$$

$$= 64\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= 64\left(\frac{5-3}{5 \cdot 3}\right)$$

$$= \frac{64 \cdot 2}{15}$$

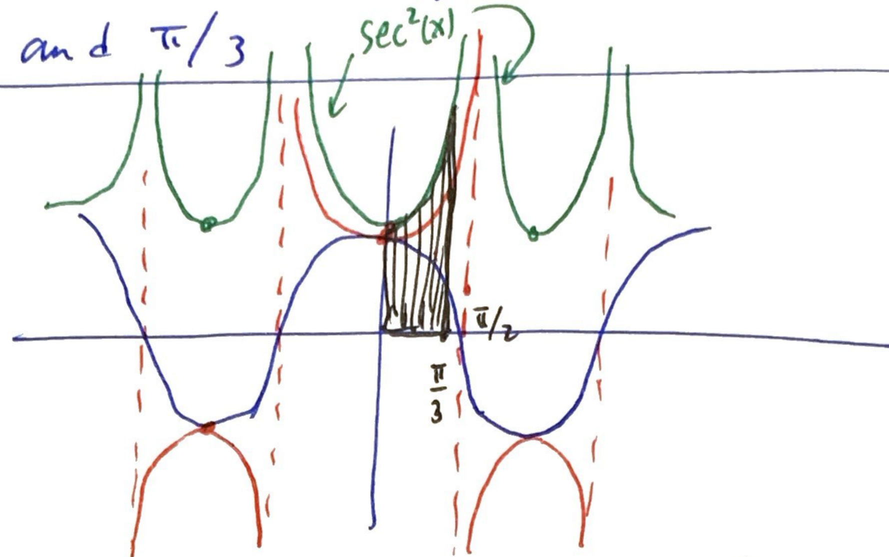
$$= \boxed{\frac{128}{15}}$$

Ans.

Ex Find the area under $y = \sec^2 x$ between 0 and $\pi/3$

Graph: $y = \sec(x)$

or $y = \dfrac{1}{\cos(x)}$

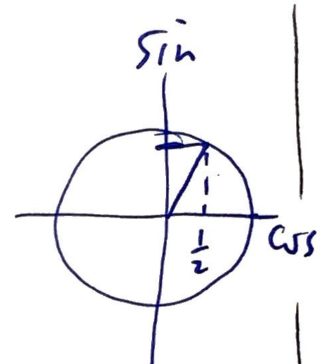


But we need $\sec^2(x)$, so we square the red curves.

Area: $\displaystyle\int_0^{\pi/3} \sec^2(x)\,dx$

FTC II we need the antiderivative of $\sec^2(x)$

i.e. $\dfrac{dF}{dx} = \sec^2(x)$



Recall from Chpt $\dfrac{d\tan(x)}{dx} = \sec^2(x)$

Then $\displaystyle\int_0^{\pi/3} \sec^2(x)\,dx = \tan\left(\frac{\pi}{3}\right) - \tan(0)$

$$= \frac{\sin(\pi/3)}{\cos(\pi/3)} - \frac{\sin(0)}{\cos(0)}$$

$$= \frac{\sqrt{3}/2}{1/2} - \frac{0}{1} = \boxed{\sqrt{3}} \text{ sq. units}$$

II Inverse Process

FTC I $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

undoes integration

FTC II $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

undoes differentiation

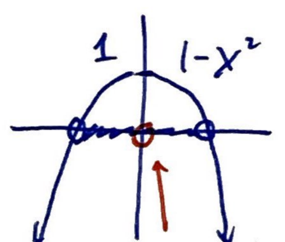
Ex Find out where $\int_0^x (1-t^2)\sin^2(t)\,dt$ is increasing?

we need to know where $\frac{d}{dx}\int_0^x (1-t^2)\sin^2(t)\,dt > 0$

so when is $(1-x^2)\sin^2(x)$ positive (By FTC I)

- $\sin^2(x)$ is always $(+)$
- $1-x^2 > 0 \rightarrow 1 > x^2$ (or $x^2 < 1$) $\quad$ $x < -1$ or $1 > x$

$-1 < x < 1$ but $x \neq 0$



exclude $x=0$

otherwise $1-x^2=0$

$\Longrightarrow$ $\boxed{(-1,0) \cup (0,1)}$ the integral increases when the upper limit $\{x \mid |x|<1 \text{ with } x \neq 0\}$