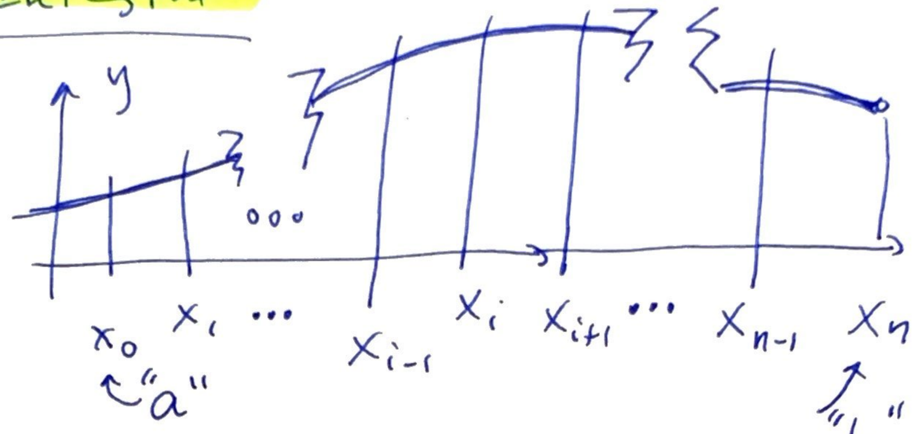
4.2 The definite Integral

we formally define the integration process

I The Definite Integral

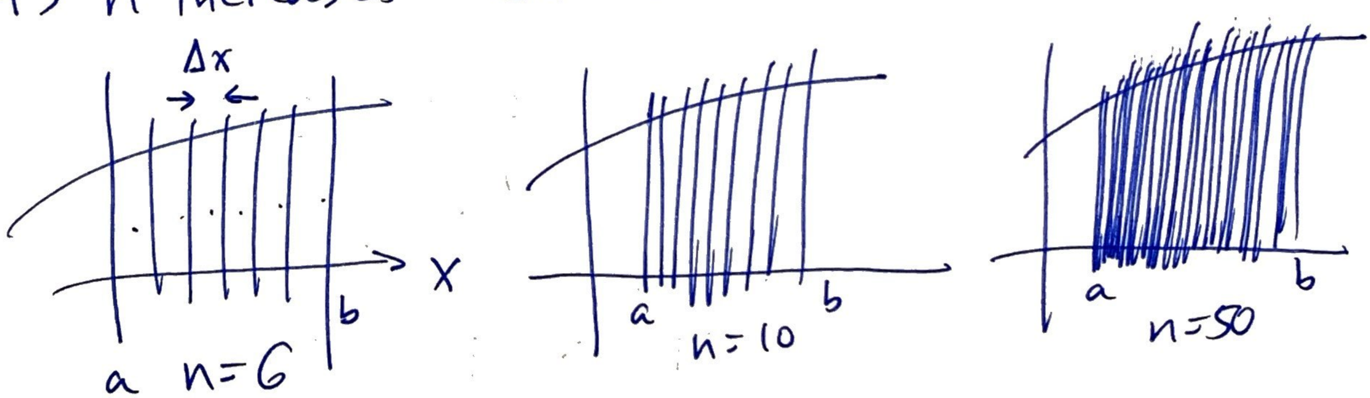
- preliminaries



- chop the interval $[a,b]$ up into n-pieces.
  width of the rectangles is

$$\boxed{\Delta x = \frac{b-a}{n}}$$

- As "n" increases Δx decreases



- we pick a sample point inside of each rectangle $x_i \leq x_i^* \leq x_{i+1}$

- form a set of sample points.

$$\{x_1^*, x_2^*, x_3^*, \dots, x_{n-1}^*, x_n$$

Definition

The **Definite integral** of $f(x)$ from "a" to "b"

is $$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$

provided that (i) the limit exists

(ii) the limit results in the same value no matter the sample set $\{x_1^*, x_2^*, \ldots\}$

- The integration sign is a stretched "S" $\int$
- If the limit exists we say that $f(x)$ is integrable on $[a,b]$
- $\int_a^b f(x)dx$ is read as "The integral from 'a' to 'b' of $f(x)$ dee x". a & b are called limits.
- $\int_a^b f(x)dx$ = a fixed number, and is NOT a variable

- The delta-epsilon explain (we skipped) reads as

For every smaller and smaller number $\varepsilon > 0$ given to us we can find a number N such that for all $n > N$ we have the inequality

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

This must hold for every choice of sample points x_i^* in the region $[x_i, x_{i+1}]$

- The proper presentation of the integral is

$$\boxed{\int_a^b \underline{\qquad}\, dx}$$

These symbols must be both expressed together

must match

The "dx" tells us that the limits of integration are constants in "x".

Bad examples:

$$\int 3x^5 \quad \text{vs.} \quad \underbrace{\int 3x^5\, \underline{dx}}_{\text{correct}}$$

• Terminology

$$\int_{x=a}^{x=b} f(x)\,dx$$

ex:

$$\int_{y=1}^{2}\left(\int_{x=\sqrt{y}}^{x=3} f(x,y)\,dx\right)dy$$

• An indefinite integral has no limits present and the results are usually a variable

indef.

$$\int x^3\,dx = \frac{x^4}{4}+c$$

or

$$\int f(x)\,dx = F(x)+c$$

but

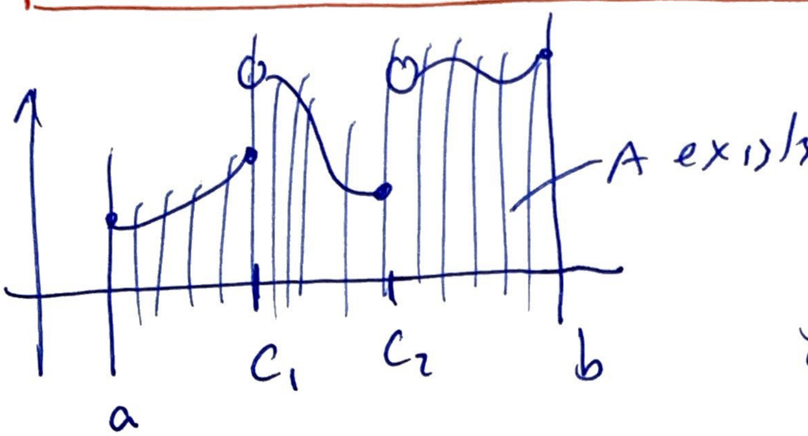
$$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \left(\frac{1^4}{4}-\frac{0^4}{4}\right) = \boxed{\frac{1}{4}}$$

definite

• The interpretation of $\int_a^b f(x)\,dx$ being the area under $f(x)$ is not entirely correct. We will see that $\int f(x)\,dx < 0$

"neg" area

Thm: If $f(x)$ is continuous on $[a,b]$ or otherwise has a finite set of jump discontinuity, then $f(x)$ is integrable on $[a,b]$ and thus $\int_a^b f(x)\,dx$ exists



$$\int_a^b f\,dx = \int_a^{c_1} f\,dx + \int_{c_1}^{c_2} f\,dx + \int_{c_2}^{b} f$$

- The Riemann Sum is the sum we used to define the integral

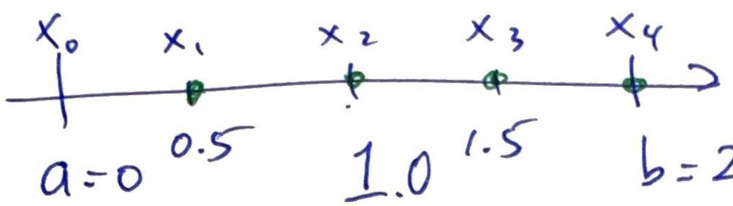
$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

$\boxed{\text{II}}$ Evaluating basic integrals

$\boxed{\text{EX}}$ let $f(x) = 2x^3 - 3x^2 + 4x - 5$

(a) evaluate a "Reimann Sum" for $f(x)$ with the RHS sampling and $a=0$, $b=2$, $n=4$

(i)



$\boxed{\text{samples } \{0.5, 1.0, 1.5, 2.0\}}$

$\Delta x = \dfrac{2-0}{4} = \boxed{1/2}$, dx

(ii)

$$RHS = \sum_{i=1}^{4} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

(iii) populate

$$= \left[2(\underline{\underline{0.5}})^3 - 3(0.5)^2 + 4(0.5) - 5\right]\cdot\frac{1}{2}$$

$$+ \left[2(\underline{\underline{1.0}})^3 - 3(1.0)^2 + 4(1.0) - 5\right]\cdot\frac{1}{2}$$

$$+ \left[2(\underline{\underline{1.5}})^3 - 3(1.5)^2 + 4(1.5) - 5\right]\cdot\frac{1}{2}$$

$$+ \left[2(\underline{\underline{2.0}})^3 - 3(2.0)^2 + 4(2.0) - 5\right]\cdot\frac{1}{2}$$

$$= \left[\frac{7}{2}\right]\frac{1}{2} + \left[-2\right]\frac{1}{2} + \left[1\right]\frac{1}{2} + \left[7\right]\frac{1}{2}$$

(iv) calculate $= \dfrac{5}{4}$ or $\boxed{1.25}$ estimate $\displaystyle\int_0^2 \left[2x^3 - 3x^2 + 4x - 5\right]dx$ w/ $n=4$

(b) fully evaluate the R.S. (Riemann Sum) for n

$$\int_0^2 (2x^3-3x^2+4x-5)\,dx$$

$$= \lim_{n\to\infty}\left\{\left[\sum_{i=1}^{n}\left[2(x_i)^3-3(x_i)^2+4(x_i)-5\right]\Delta x\right\}$$

Here $x_i = i\Delta x$ & $\Delta x = \frac{2-0}{n}$ (end, beg.) $= \frac{2}{n} \Rightarrow \boxed{x_i = i\frac{2}{n}}$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\underbrace{\left[2\left(\frac{2i}{n}\right)^3-3\left(\frac{2i}{n}\right)^2+4\left(\frac{2i}{n}\right)-5\right]}_{f(x_i)}\underbrace{\left(\frac{2}{n}\right)}_{\Delta x}\right\}$$

$$= \lim_{n\to\infty}\left\{\left[\sum_{i=1}^{n}\frac{2\cdot 2^3}{n^3}\cdot i^3-\sum_{i=1}^{n}\frac{3\cdot 2^2}{n^2}i^2+\sum_{i=1}^{n}\frac{4\cdot 2}{n}\cdot i-\sum_{i=1}^{n}5\right]\frac{2}{n}\right.$$

$$= \lim_{n\to\infty}\left\{\frac{32}{n^4}\left(\sum_{i=1}^{n}i^3\right)-\frac{24}{n^3}\left(\sum_{i=1}^{n}i^2\right)+\frac{16}{n^2}\left(\sum_{i=1}^{n}i\right)-\frac{2\cdot 5}{n}\left(\sum_{i=1}^{n}1\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{32}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)-\frac{24}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+\frac{16}{n^2}\left[\frac{n(n+1)}{2}\right]-\frac{10}{\cancel{n}}(\cancel{n})\right\}$$

$$= \lim_{n\to\infty}\left\{8\frac{n^2}{n^2}\left(\frac{n}{n}+\frac{1}{n}\right)^2-4\left(\frac{n}{n}\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right)\right)+8\left(\frac{n}{n}\left(\frac{n}{n}+\frac{1}{n}\right)\right)-10\right\}$$

$$= \lim_{n\to\infty}\left\{8\cdot 1\cdot\left(1+\cancelto{0}{\tfrac{1}{n}}\right)-4\left(1+\cancelto{0}{\tfrac{1}{n}}\right)\left(2+\cancelto{0}{\tfrac{1}{n}}\right)+8\left(1+\cancelto{0}{\tfrac{1}{n}}\right)-10\right\}$$

$$= 8\cdot 1-4\cdot 1\cdot 2+8\cdot 1-10 = \boxed{-2}$$ exact answer.

Note: we will present formulas, such as

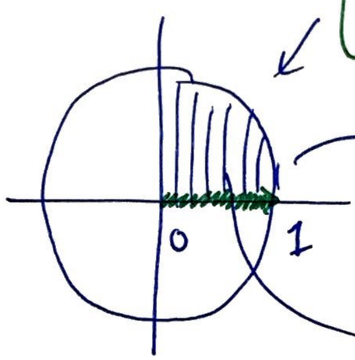
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad \frac{d\left(\frac{x^{n+1}}{n+1} + c\right)}{dx} = x^n$$

Note: we need not use formulas or the summation formulas if we can interpret the integral via geometry.

$\boxed{\text{EX}}$

$$\int_0^1 \overbrace{\sqrt{1-x^2}}^{y=}\,dx \quad \text{using area of a circle}$$

$\boxed{x^2+y^2=1}$



$y = \sqrt{1-x^2}$

$$\text{area} = \left.\frac{\pi r^2}{4}\right|_{r=1} = \frac{\pi}{4}$$

So

$$\boxed{\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}}$$

$\boxed{\text{III}}$ Numerical Integration (Introduction)

If we cannot analytically evaluate an integral we can use rectangle to approximate it.

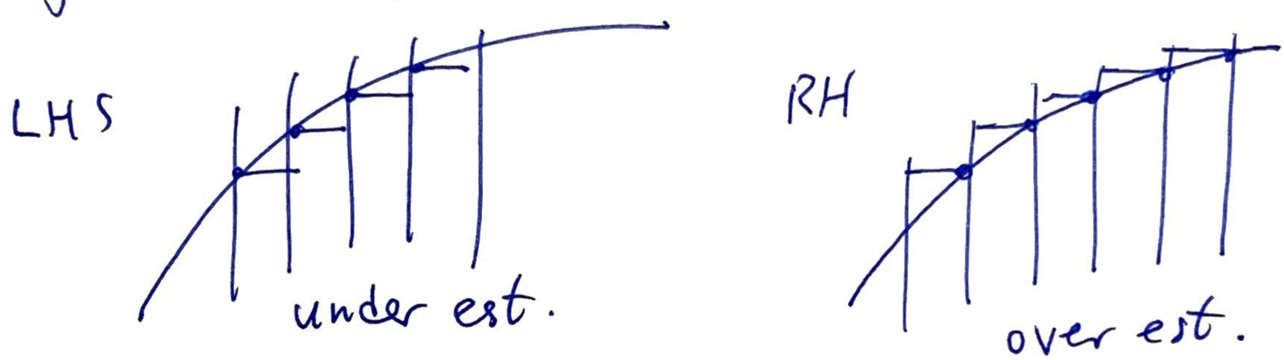
The MidPoint Rule

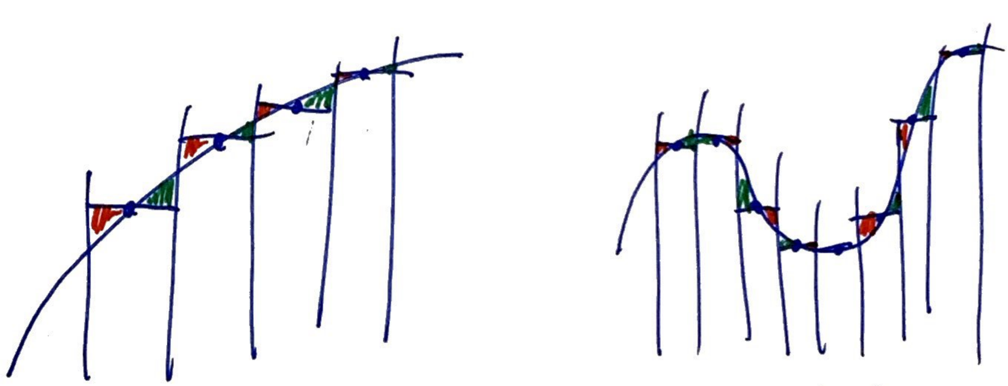
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\bar{x}_i$ is the midpoint of the rectangle

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} \text{ in the intvl } [x_{i-1}, x_i]$$

Why the midpoint?





The over estimation and under estimation "cancels out" generally speaking.

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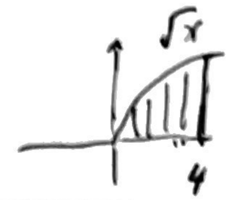
1. Use the <u>midpoint rule</u> to approximate the integral $\int_0^4 \sqrt{x}\,dx$, $n=8$



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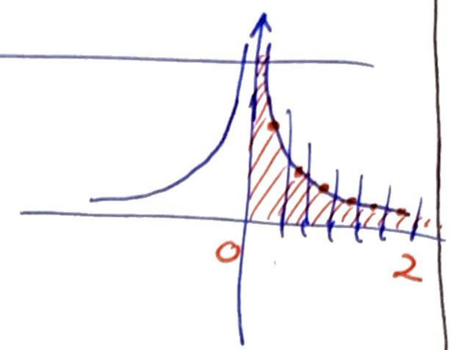
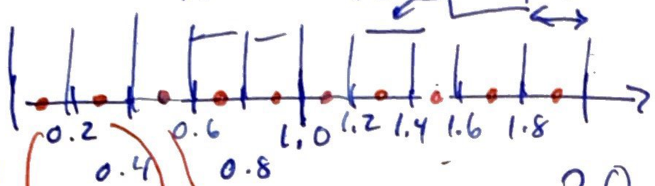
Ex Use the midpoint rule with $n=10$ to approximate the integral $\displaystyle\int_{\text{"}x=0\text{"}}^{x=2} \frac{1}{x^2}\,dx$



(i) $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \boxed{\dfrac{1}{5}}$

(ii)



$\{$

$\{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5,\ 1.7,\ 1.9\}$

(iii) $\displaystyle\int_0^2 \frac{dx}{x^2} \approx f(0.1)\cdot\frac{1}{5} + f(0.3)\frac{1}{5} + f(0.5)\frac{1}{5} + \cdots + f(1.7)\frac{1}{5} + \frac{f(1.9)}{5}$

$\approx \left(\dfrac{1}{0.1^2}\right)\dfrac{1}{5} + \left(\dfrac{1}{0.3}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{0.5}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{0.7}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{0.9}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{1.1}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{1.3}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{1.5}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{1.7}\right)^2\dfrac{1}{5} + \left(\dfrac{1}{1.9}\right)^2\dfrac{1}{5}$

factor out $1/5$

$\approx \left[\underset{1}{\dfrac{1}{0.1^2}} + \underset{2}{\dfrac{1}{0.3^2}} + \underset{3}{\dfrac{1}{0.5^2}} + \underset{4}{\dfrac{1}{0.7^2}} + \underset{5}{\dfrac{1}{0.9^2}} + \underset{6}{\dfrac{1}{1.1^2}} + \underset{7}{\dfrac{1}{1.3^2}} + \underset{8}{\dfrac{1}{1.5^2}} + \underset{9}{\dfrac{1}{1.7^2}} + \underset{10}{\dfrac{1}{1.9^2}}\right]\cdot\dfrac{1}{5}$

(iv) TI-30Xa $\boxed{\text{2nd}}$ $\boxed{\text{CSR}}$ 0.1 $\boxed{1/x}$ $\boxed{x^2}$ $\boxed{\Sigma+}$ 0.3 $\boxed{\frac{1}{x}}$ $\boxed{x^2}$ $\boxed{\Sigma+}$

... 1.9 $\boxed{\frac{1}{x}}$ $\boxed{x^2}$ $\boxed{\Sigma+}$

$\boxed{\text{2nd}}$ $\boxed{\Sigma x}$, $[120.8721311]\cdot\dfrac{1}{5} = \underline{\underline{24.174426}}$

Check on Desmos →

IV Properties of Definite Integrals

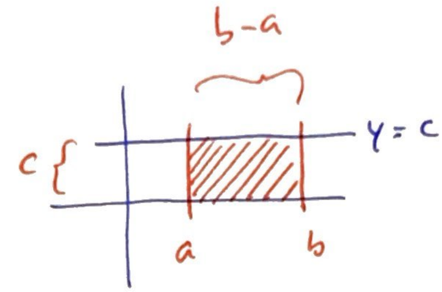
$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

accumulation is direction dependent.

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$$\boxed{\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx}$$

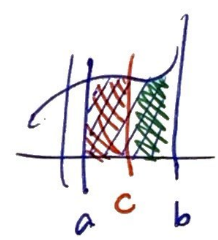
$$\boxed{\int_a^b c\,dx = c(b-a)}$$



$$\boxed{\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx}$$

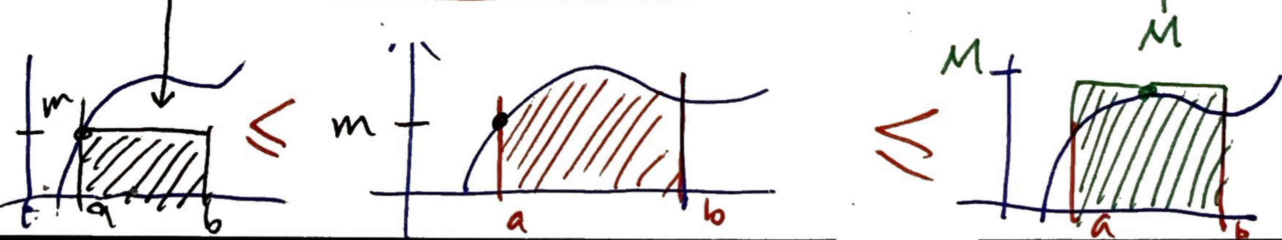
$$\boxed{\int_a^b (f-g)\,dx = \int_a^b f\,dx - \int_a^b g\,dx}$$

$$\boxed{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx}$$

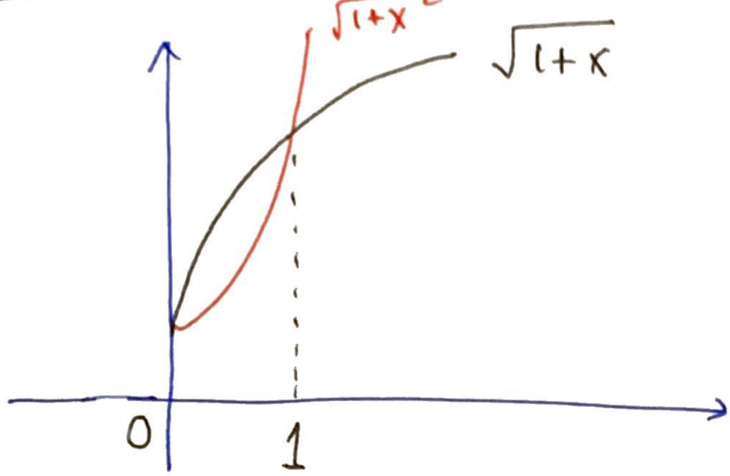


- Let M be the maximum of $f(x)$ on $[a,b]$
  Let m be the minimum of $f(x)$ on $[a,b]$

$$\boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$$



EX Verify $\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$



we see $\sqrt{1+x}$ is the greater function between $[0,1]$

I.E. $\sqrt{1+x^2} \leq \sqrt{x+1}$ visually

But analytically we note $x^2 \leq x$ on $[0,1]$

- Add 1 to both sides: $x^2 \leq x$

$$x^2+1 \leq x+1 \quad \text{still true}$$

- Square root

$$\sqrt{x^2+1} \leq \sqrt{x+1} \quad \text{still true.}$$

test

$$0.6 \overset{?}{\leq} 0.7$$
$$\sqrt{0.6} \overset{?}{\leq} \sqrt{0.7}$$
$$0.7746 \overset{?}{\leq} 0.83660 \checkmark$$

- Integrate $\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$