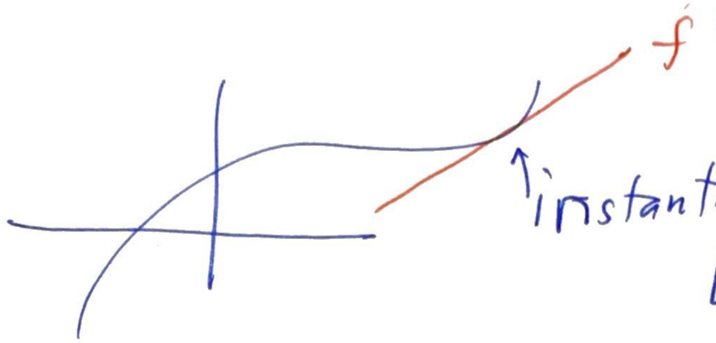
Chapter 4 Integration

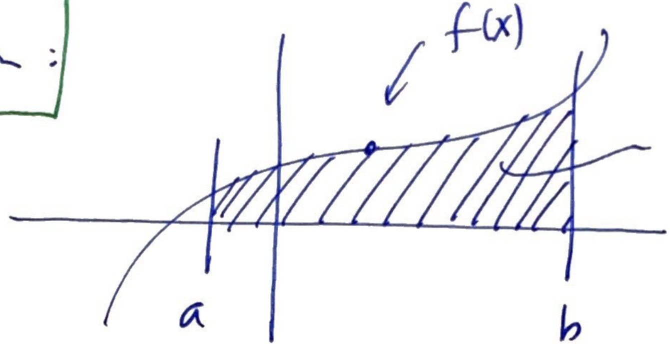
Calculus is largely differentiation (rates of change) and Integration (which is accumulation).

- Diff'n



instantaneous rate of change.

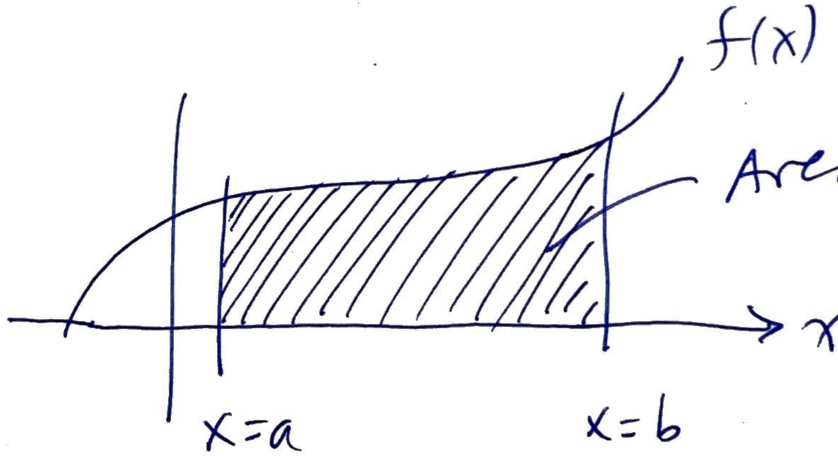
- Integration:



we accumulate all values of f(x) in some interval

4.1 Area and distance

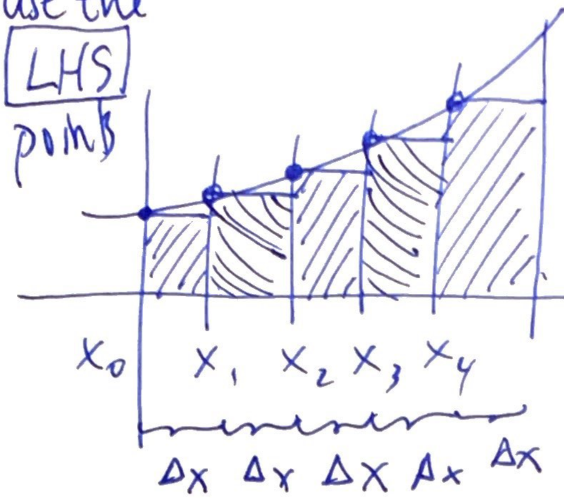
I Area under a curve



Area = ?

- Strategy is to break up the region into vertical strips and add the strips up.

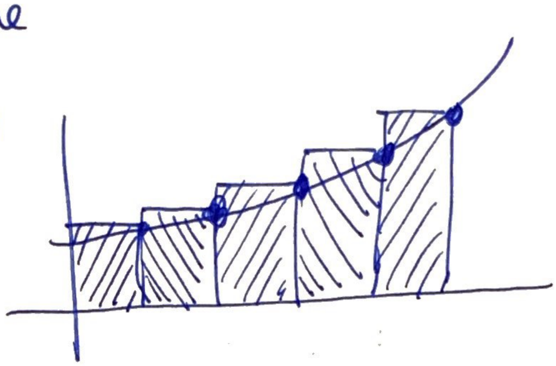
• use the $\boxed{\text{LHS}}$ points



$$\boxed{\text{Area} \approx \sum \overset{H}{f(x_0)} \cdot \overset{W}{\Delta x}}$$

if concave up LHS pts will underestimate the area.

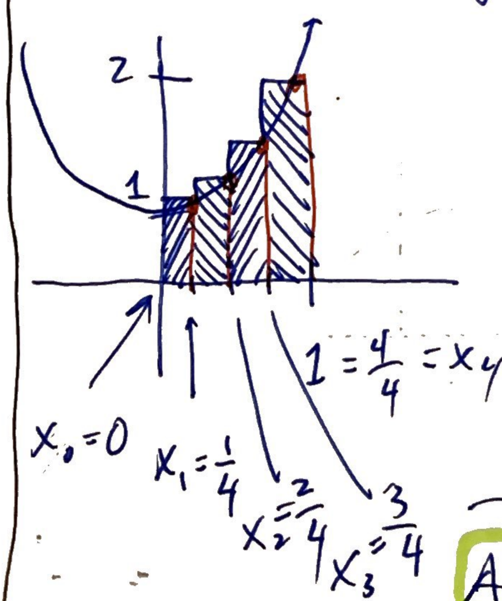
• use the $\boxed{\text{RHS}}$ points



if concave up RHS pts will overestimate the area

$\boxed{\text{Ex}}$ consider $y = x^2+1$ divide the region of $[0,1]$ into 4 rectangles.

(a) approx using the RHS of each Rectangle



$\boxed{\Delta x = 1/4} \Leftarrow \dfrac{(1-0)}{4}$

$f(x_1) = f(\frac{1}{4}) = (\frac{1}{4})^2+1 = \frac{17}{16}$

$f(x_2) = f(\frac{2}{4}) = (\frac{2}{4})^2+1 = \frac{20}{16}$

$f(x_3) = f(\frac{3}{4}) = (\frac{3}{4})^2+1 = 25/16$

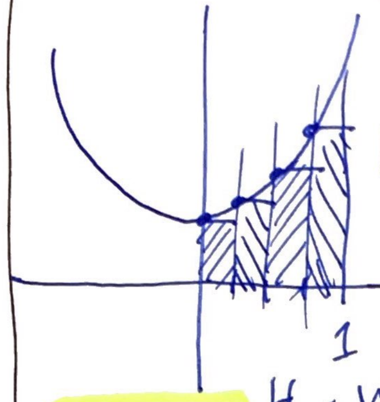
$f(x_4) = f(1) = 1^2+1 = 2 = \frac{32}{16}$

$A \approx \frac{17}{16}\cdot\frac{1}{4} + \frac{20}{16}\cdot\frac{1}{4} + \frac{25}{16}\cdot\frac{1}{4} + \frac{32}{16}\cdot\frac{1}{4} = \frac{1}{4\cdot 16}(17+20+25+32)$

$\boxed{A = 47/32}$ sq. units.

(b) now use L.H. points

W: $\boxed{\Delta x = 1/4}$



H: $f(x_0) = f(0) = 0^2+1 = 1$

$f(x_1) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = \frac{17}{16}$

$f(x_2) = f\left(\frac{2}{4}\right) = \left(\frac{2}{4}\right)^2 + 1 = \frac{20}{16}$

$f(x_3) = f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2 + 1 = \frac{25}{16}$

$$A \approx \overset{H\cdot W}{f(x_0)\cdot\Delta x} + \overset{H\cdot W}{f(x_1)\cdot\Delta x} + \overset{H\cdot W}{f(x_2)\cdot\Delta x} + \overset{H\cdot W}{f(x_3)\cdot\Delta x}$$

$$\approx \underset{\frac{16}{16}}{1}\cdot\frac{1}{4} + \frac{17}{16}\cdot\frac{1}{4} + \frac{20}{16}\cdot\frac{1}{4} + \frac{25}{16}\cdot\frac{1}{4}$$

$$\approx \frac{1}{16\cdot 4}\left[16+17+20+25\right]$$

$$\approx \frac{78}{16\cdot 4} = \boxed{\frac{39}{32}}$$

<u>Summary</u>:
- RH pts $\rightarrow \frac{47}{32} = 1.46875$
- LH pts $\rightarrow \frac{39}{32} = 1.21875$

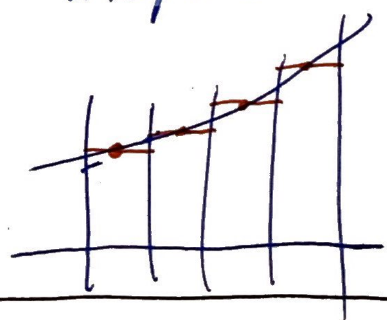
actual / exact: $\left(\frac{x^3}{3}+x\right)\Big|_0^1 = \frac{1}{3}+1 = \boxed{\frac{4}{3}} \approx \frac{40}{30} = 1.3333$

"Hey, lets average RH & LH results" aka. the Midpoint Rule

- ave $= \left(\frac{47+39}{32}\right)/2$

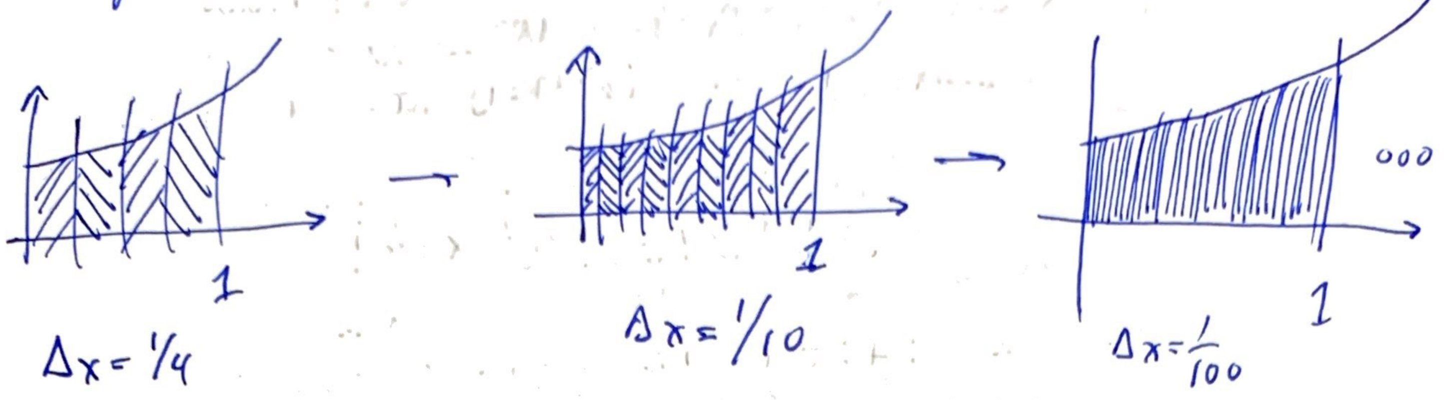
$$= \frac{86}{2\cdot 32} = \frac{43}{32} = 1.34375$$

one place acc'y



Riemann Sum

To get the exact area we let $\Delta x \to 0$



- Integral Def: $\boxed{I = \lim_{\Delta x \to 0} \sum f(x) \cdot \Delta x}$ exact.

$$\boxed{\lim_{N \to \infty} \sum_{n=1}^{N} f(x_n) \cdot \underbrace{\frac{(b-a)}{N}}_{\Delta x}}$$

N = # of strips.

- Notation

$$\boxed{\int_{x=a}^{x=b} f(x)\,dx \equiv \lim_{\Delta x \to 0} \sum \underbrace{f(x)}_{(x_0 + n\Delta x)} \cdot \Delta x} \quad \boxed{4.2}$$

"integral"

∞ many of ∞-ly thin strips

$0 + 0 + 0 + \cdots$

The Mechanics (Details) to be discussed soon.

Recall

Diff'n: $f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) \to$ "$\frac{0}{0}$"

= finite ratio

* We need to review some sums so we can evaluate the Area under curves:

From Pre-Calculus: "proved by induction" in Pre-calc.

$$\boxed{1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}}$$

ex: sum ~~the~~ squares of the 1st 12 integers

$$1^2+2^2+3^2+\cdots+11^2+12^2 = \frac{12(12+1)(2\cdot 12+1)}{6}$$

$$= \frac{\cancel{12}^{\,2}\cdot 13\cdot 25}{\cancel{6}}$$

$$= 26\cdot 25$$

$$= 600+2\cdot 25 = 650$$

Ex: Back to the example but now use "n" strips vs. 4.

Then the approx of area using the RHS approach

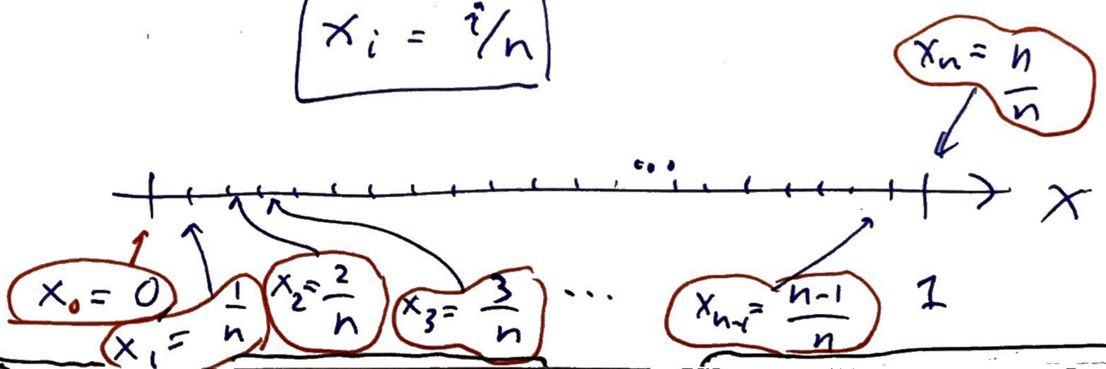
RHS

$$\boxed{R_n = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_n)\cdot\Delta x}$$

sum of n regions

where $x_i = \underbrace{x_1}_{0} + i\cdot\Delta x = 0 + i\,\frac{(\overset{1}{b}-\overset{0}{a})}{n}$

$$\boxed{x_i = i/n}$$



$x_i = \frac{i}{n}$ $\qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

Then $f(x_i) = x_i^2 + 1$

becomes $\boxed{f_i = \left(\frac{i}{n}\right)^2 + 1}$

last strip's area

$$\Rightarrow R_n = \underbrace{f(x_1)\cdot\Delta x}_{A_1} + \underbrace{f(x_2)\cdot\Delta x}_{A_2} + \cdots \underbrace{f(x_n)\cdot\Delta x}_{A_n}$$

so $\boxed{R_n = \left[\left(\frac{1}{n}\right)^2 + 1\right]\left(\frac{1-0}{n}\right) + \left[\left(\frac{2}{n}\right)^2 + 1\right]\left(\frac{1-0}{n}\right) + \left[\left(\frac{3}{n}\right)^2 + 1\right]\frac{1}{n} + \cdots + \left[\left(\frac{n}{n}\right)^2 + 1\right]\cdot\frac{1}{n}}$

$$= \left(\frac{1}{n}\right)\cdot\left[\left[\left(\frac{1}{n}\right)^2 + 1\right] + \left[\left(\frac{2}{n}\right)^2 + 1\right] + \left[\left(\frac{3}{n}\right)^2 + 1\right] + \cdots + \left[\left(\frac{n}{n}\right)^2 + 1\right]\right]$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots \left(\frac{n}{n}\right)^2 + n\cdot 1\right]$$

$$= \frac{1}{n}\left[\frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^2} + n\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)/6}{n^2} + n\right]$$

$$\boxed{R_n = \frac{n(n+1)(2n+1)}{6n^3} + 1}$$

RH pts

Now let $n=4$

$$R_4 = \frac{4(4+1)(2\cdot 4+1)}{6\cdot 4^3} + 1$$

$$= \frac{\cancel{4}\cdot 5\cdot 9}{6\cdot 4^2\cdot\cancel{4}} + 1 = \frac{45}{96} + 1$$

$$= \frac{45+96}{96} = \frac{141}{96} = 1.46875$$

as we already calculated...

What about $n = 100$?

$$\bullet\ R_{100} = \frac{\cancel{100}(101)(201)}{6\cdot 100^2\cdot\cancel{100}} + 1$$

$$= 1.338335$$

$n=1000$

$$\bullet\ R_{1000} = \frac{\cancel{1000}(1001)(2001)}{6\cdot 1000^2\cdot\cancel{1000}} + 1 = 1.3338335$$

(*) What if $n \to \infty$

exact area is $\lim_{n\to\infty} R_n$

$$A = \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3} + 1\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n}\right)\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right)\cdot\frac{1}{6} + 1\right]$$

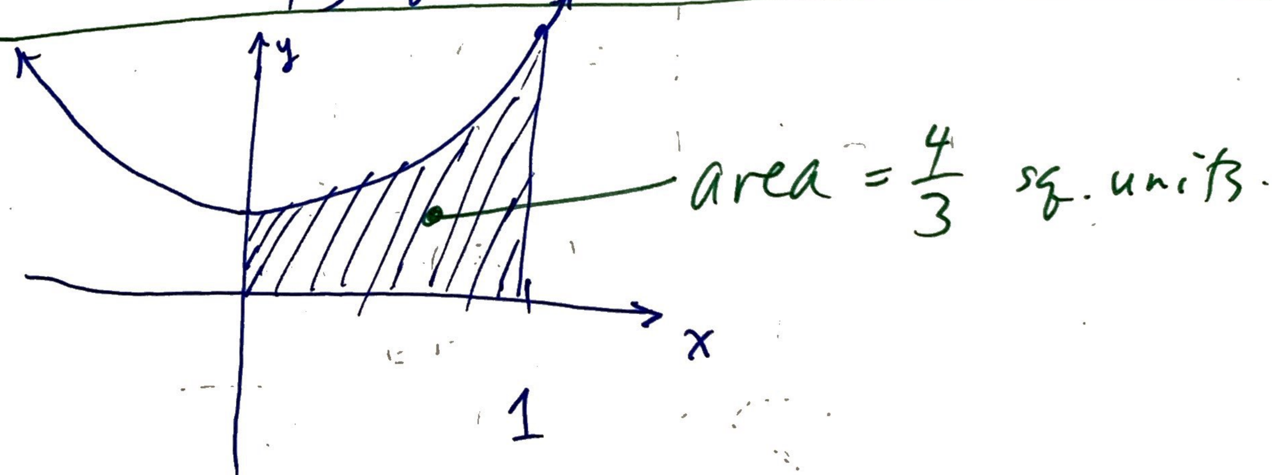
$$= \lim_{n\to\infty}\left[(1)\cdot\left(1+\cancelto{0}{\frac{1}{n}}\right)\cdot\left(2+\cancelto{0}{\frac{1}{n}}\right)\cdot\frac{1}{6}+1\right]$$

$$= \frac{1\cdot 1\cdot 2}{6} + 1$$

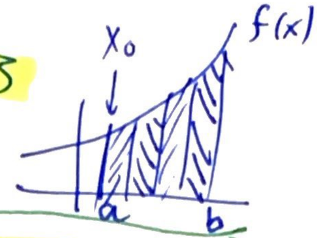
$$= \frac{2}{6}+1 = \frac{1}{3}+1 = \boxed{\frac{4}{3}}$$

exact answer for ∞ many strips

The exact area under $f(x) = x^2+1$ from $x = 0$ to 1 is $4/3$ sq. units



General procedures and comments



- Width of strips (rectangles) for n-rectangles

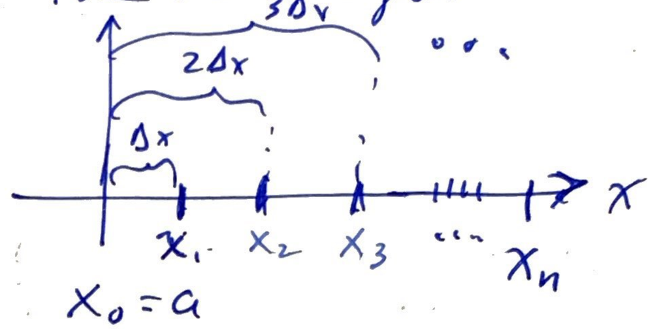
$$\boxed{\Delta x = \frac{b-a}{n}}$$

- Subintervals for each strip:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$$

- locations of the sides of the rectangles

$$\begin{cases} x_0 = \text{beginning} = a \\ x_1 = a + \Delta x \\ x_2 = a + 2\Delta x \\ \vdots \\ x_n = a + n\Delta x \end{cases}$$



- height $= f(x_i)$
- $\text{area}_i = \text{height}_i \cdot \Delta x = f(x_i) \cdot \Delta x$

RH Sum: $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\cdot\Delta x$

LH Sum: $L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\cdot\Delta x$

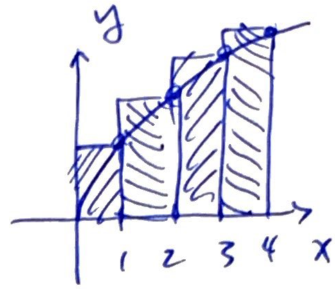
Def: The Accumulation of $f(x)$ in the region $x=a$ to $x=b$, assuming $f(x)$ is continuous, is

$$A = \lim_{n\to\infty}(R_n) = \lim_{n\to\infty} \sum_{n=1}^{n} f(x_i)\cdot\Delta x$$

Math 211 CW 4a Name ______

1. let $f(x) = \sqrt{x}$, Estimate the area from $x = 0 + 4$

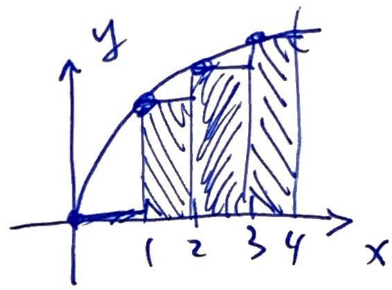
(a) using the "RHS points", with $n = 4 \Rightarrow \boxed{\Delta x = 1}$



$$R_4 = \overbrace{\sqrt{1}\cdot 1 + \sqrt{2}\cdot 1 + \sqrt{3}\cdot 1 + \sqrt{4}\cdot 1}^{f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x + f(4)\cdot\Delta X}$$

$\approx \underline{\underline{6.146}}$

(b) using the LHS points



$L_4 = f(0)\cdot\Delta x + f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x$

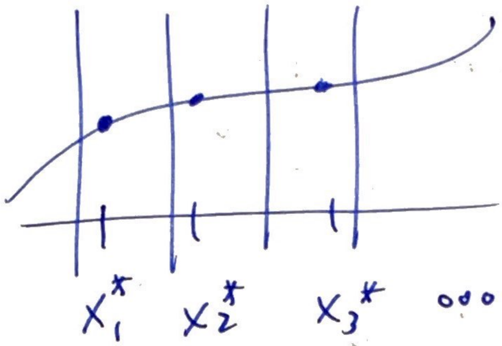
$= \sqrt{0}\cdot 1 + \sqrt{1}\cdot 1 + \sqrt{2}\cdot 1 + \sqrt{3}\cdot 1$

$= 4.146$

(c) state an upper and lower bounds for the exact ans: $\underbrace{4.146} < A < \underbrace{6.146}$

- more generally we can evaluate $f(x)$ in the <u>i^{th} interval</u> at <u>any</u> x_i^* in the interval $[x_i, x_{i+1}]$

$$x_i \leq x_i^* \leq x_{i+1}$$



⊛ More sum tools :

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

5 y.o.

Ex Sum the integers from 1 to 100: Euler's method

$1+2+3+\cdots+48+49+50+51+52+\cdots+97+98+99+100$

100

100

100

100

100

100

Sum $= 49 \cdot 100 + 50 + 100$

$4900 + 150$

$= \boxed{5050}$

$5000 + 50$

5050

• Test with sum:

$$\sum_{1}^{100} i = \frac{100(100+1)}{2} = \frac{10100}{2}$$

- $$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

- $$\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$$

$$a \pm b \begin{cases} a+b \\ a-b \end{cases}$$

Recall also

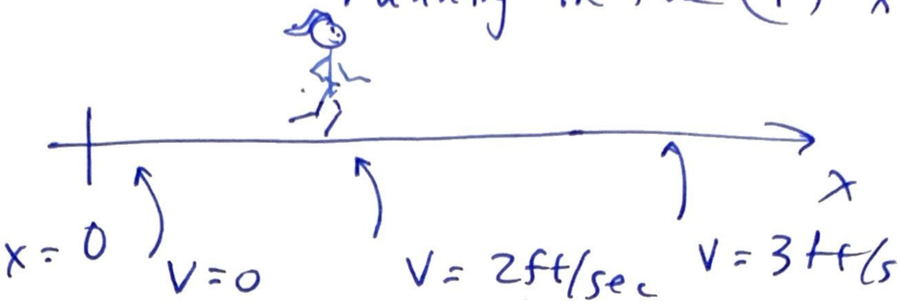
- $\displaystyle\sum_{i=1}^{n} c = n \cdot c$
- $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
- $\displaystyle\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

* These will be use more extensively in 4.2 and beyond. *

We will turn to some applications!

II Distance

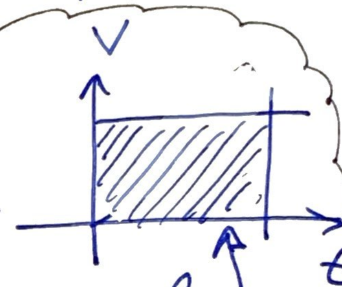
Consider a runner running in the (+) x-direction



They run faster and faster.

Their distance can be estimated if at each Δx we calculate the speed.

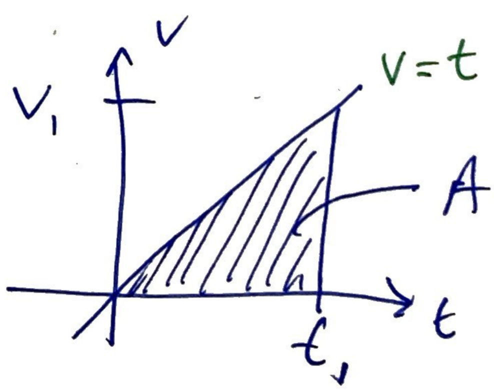
Recall $\boxed{D = V \cdot t}$ • V = const.



"Distance is area" = distance covered

If $V \neq$ const, for example $\boxed{V(t) = t}$

then D = area under the curve

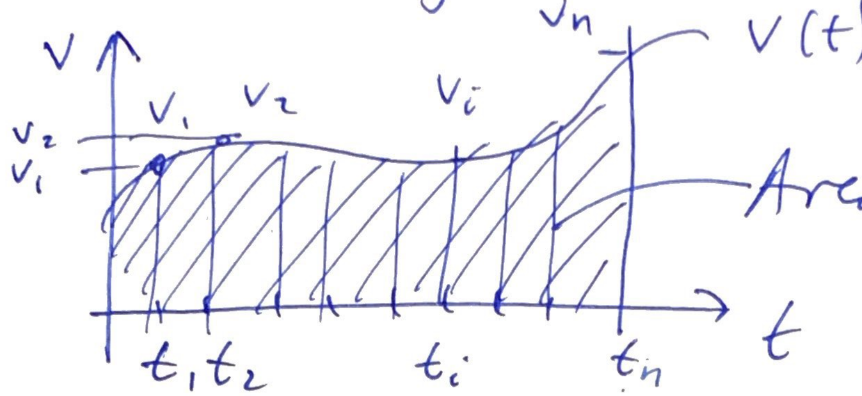


Area $= \frac{1}{2}$ base · height

$= \frac{1}{2} t_1 \cdot V_1$

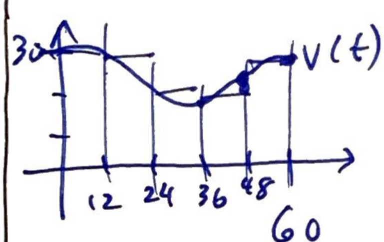
$$D = \frac{1}{2} V t$$

- For V = variable velocity then distance covered is just the area under the $V(t)$ curve.



$$\boxed{\text{approx. distance} = \sum_{i=1}^{n} V(t_i)\cdot\Delta t}$$

Ex We tabulate experimentally determined data from the speed of the runner



t s	0	12	24	36	48	60
V in/s	30	28	25	22	24	27

Q: Estimate the distance covered:

$$D \approx V_1\cdot\Delta t + V_2\cdot\Delta t + V_3\cdot\Delta t + V_4\cdot\Delta t + V_5\cdot\Delta t$$

LHS estimate: $30\cdot 12s + 28\cdot 12s + 25\cdot 12s + 22\cdot 12s + 24\cdot 12s$

$= 1548$ inches/sec

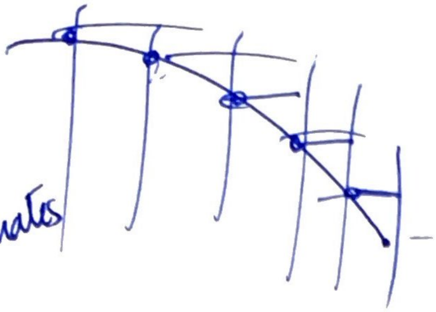
RHS estimate: $= 28\cdot 12s + 25\cdot 12s + 22\cdot 12s + 24\cdot 12s + 27\cdot 12s$

$= 1512$ inches/sec

Note since $V(t)$ is not monotonic we cannot estimate via a LHS and RHS amount.

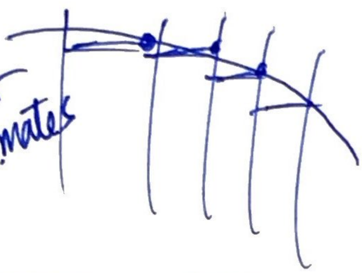
LH

over estimates



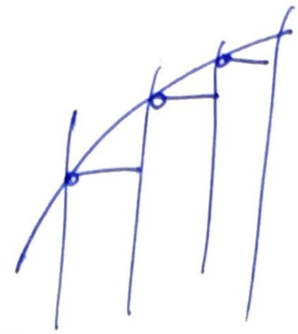
RH

Under estimates



$$\boxed{A_{RHS} < A < A_{LHS}}$$

monotonic increasing

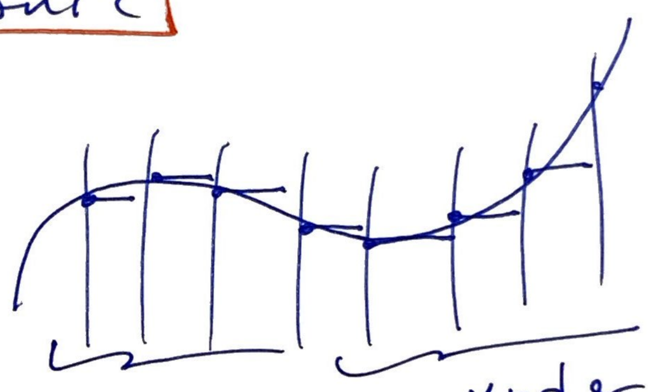


LH Underestimates

RH overestimates

$$\boxed{A_{LHS} < A < A_{RHS}}$$

- Non-monotonic

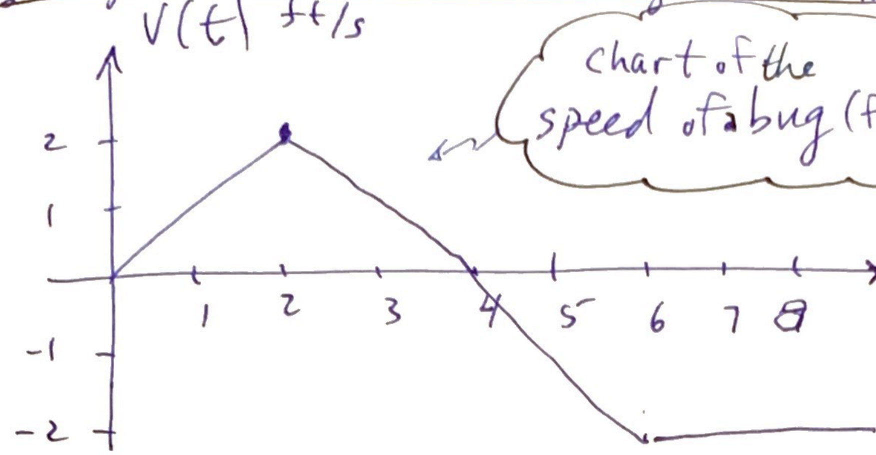


over est   under est.

- we cannot use LHS and RHS as error bounds...
- we could average to get a "better" answer as this would basically be the midpoint rule.

(*) using a graph of $v(t)$ to compute distance.

EX



(+)

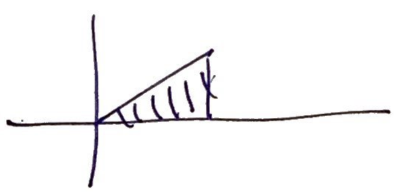
chart of the speed of a bug (fast!)

1-Dim motion

(a) given the above chart, at what time does the bug reverse direction? $\boxed{v(t) @ t = 4s}$

(b) after 2 sec how far has the bug travelled

$d = v \cdot t$ = area under curve between 0 & 2s



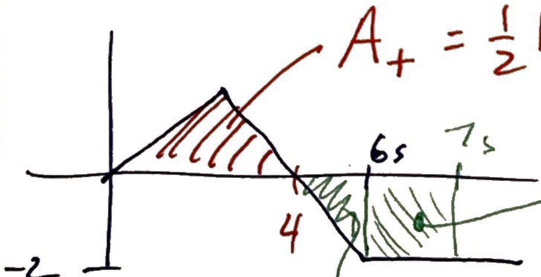
$d = \frac{1}{2}$ base $\cdot$ height

$= \frac{1}{2}(2 - 0s) \cdot (2 \, ft/s) = \boxed{2 ft}$

(c) When (time) has the bug reached the starting point?

Here we need to know when (+) area is equal to (−) area.

$A_+ = \frac{1}{2} b \cdot h = \frac{1}{2}(4 - 0_s) \cdot 2 \frac{ft}{s} = \underline{\underline{4 ft}}$.



$A = -2 ft$

$A_- = \frac{1}{2} b \cdot h = \frac{1}{2}(6 - 4_s)(-2 \frac{ft}{s}) = -2$ ft back @ $t = 6s$

we need another ...
−2ft
So ... $= 7s$