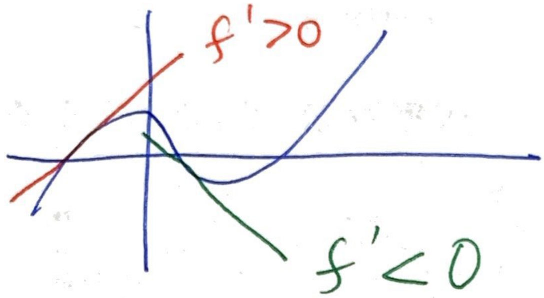
3.3 Using Derivatives to Determine the Shapes of Curves

I what does f' say about f ?

- If $f'(x) > 0$ then f is increasing
  If $f'(x) < 0$ then f is decreasing

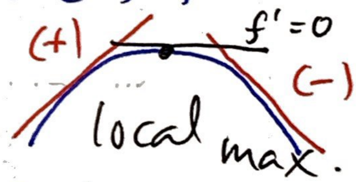


The derivative Test:

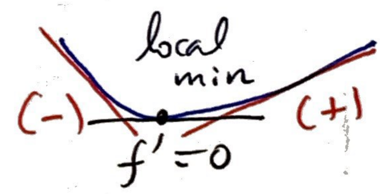
Suppose "c" is a critical value.

i.e. $f'(c) = 0$ or $f'(c)$ is undefined or $f'(c)$ D.N.E.

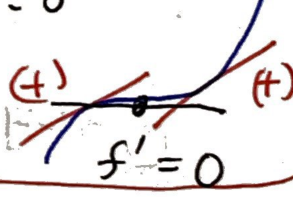
(a) If f' changes sign from (+) to (−) then $f(c)$ is a local max.



(b) If f' changes from (−) to (+) then $f(c)$ is a local min



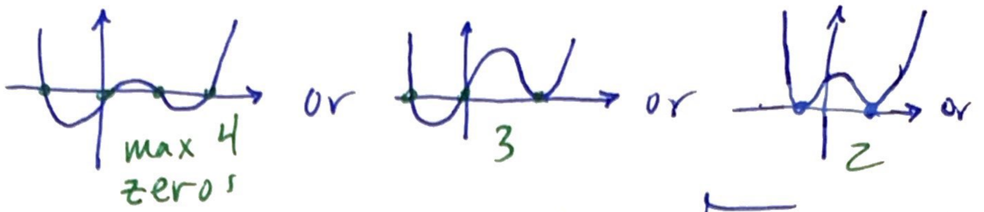
(c) If f' around "c" does not change sign f has no local max or min



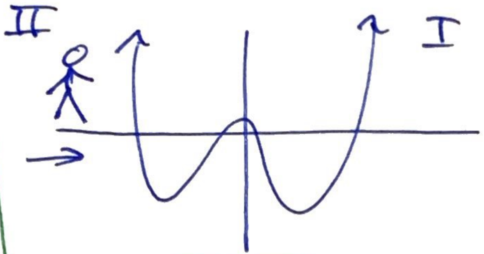
EX

Sketch $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

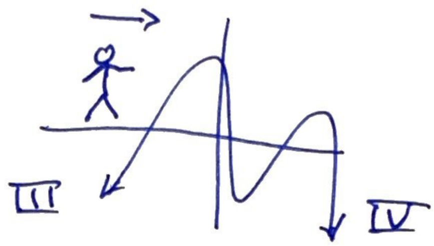
x^4: "W"



• "Even" power so we come from quadrant $\boxed{\text{II}}$ and leave in quadrant $\boxed{\text{I}}$ since a_4 on $a_4x^4 + a_3x^3 \ldots$ is positive



$\boxed{a_4 > 0}$ (+)



$\boxed{a_4 < 0}$ (−)

• Derivative Test:

$f'(x) = 12x^3 - 12x^2 - 24x$

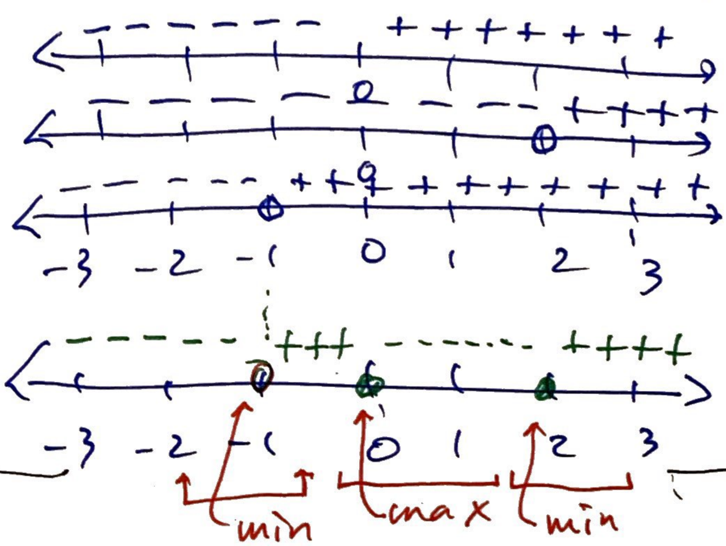
$= 12x(x^2 - x - 2)$

$\boxed{f'(x) = 12x(x-2)(x+1)}$

• $\boxed{\text{Sign analysis}}$ {via # Lines or Test Points}

Individual: x, $x-2$, $x+1$

product: $x(x-2)(x+1)$

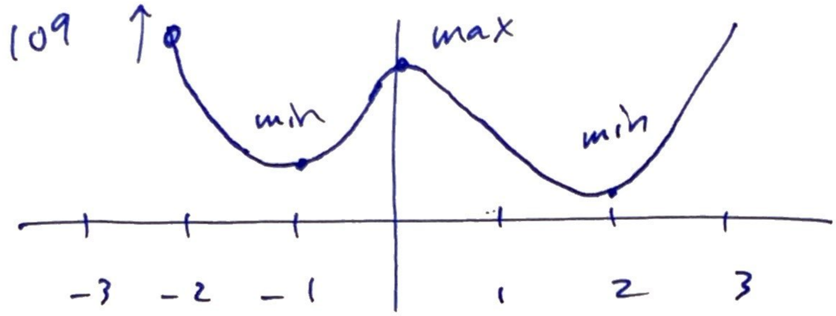


EX (cont.)

• helper points: pick $x < -1$, like "-2" say

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 5$

$= 48 + 32 + 24 + 5 = \underline{109}$



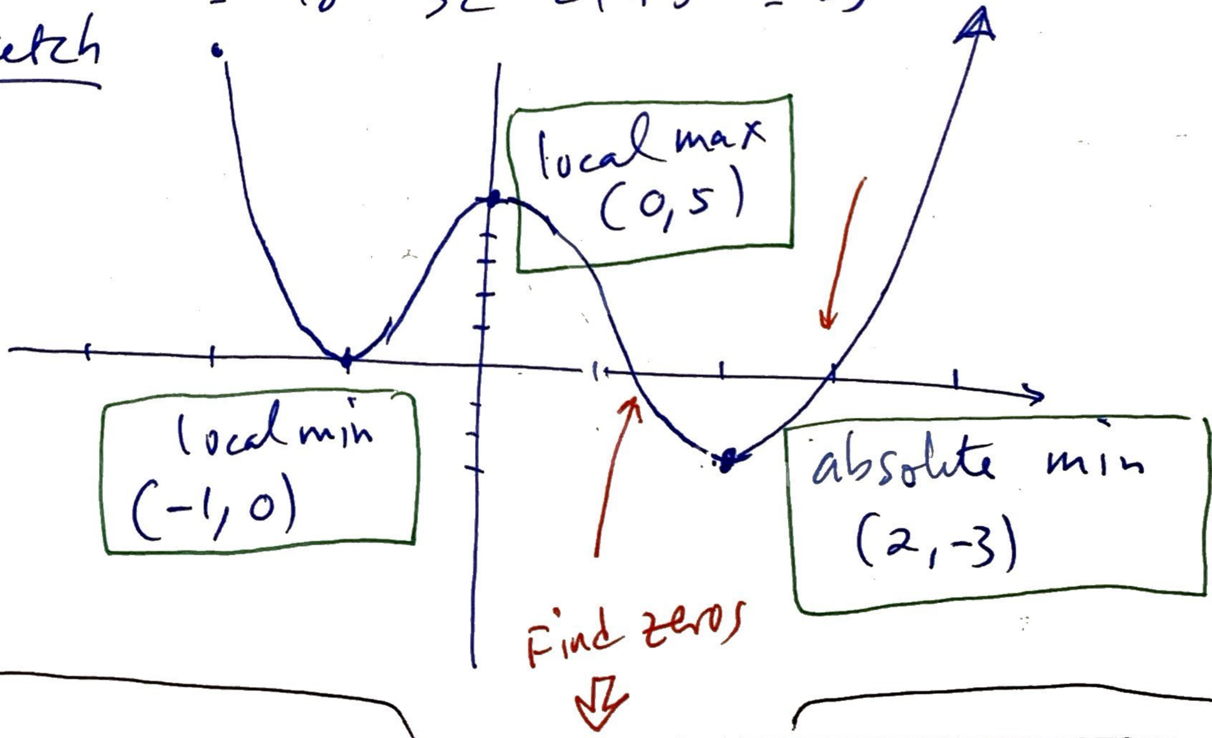
• more helper points @ critical values:

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5$
$= 3 + 4 - 12 + 5 = 0$

$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 5 = 5$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5 =$
$= 48 - 32 - 24 + 5 = -3$

• sketch



EX (cont.)

- "Finding Zeros": mymathmantra.com → Diff Eqn Reserve

We know $x=-1$ is a zero.

$$\begin{array}{r|rrrrr} & 4 & 3 & 2 & 1 & 0 \\ -1 & 3 & -4 & -12 & 0 & 5 \\ & \downarrow & -3 & 7 & 5 & -5 \\ \hline & 3 & -7 & -5 & 5 & 0 \end{array}$$

- $3x^4-4x^3-12x^2+5 = (3x^3-7x^2-5x+5)(x+1)$

$$\begin{array}{r|rrrr} -1 & 3 & -7 & -5 & 5 \\ & & -3 & 10 & -5 \\ \hline & 3 & -10 & 5 & 0 \end{array}$$

- $3x^4-4x^3-12x^2+5 = \underline{\underline{(3x^2-10x+5)}}(x+1)(x+1)$

$\frac{p}{q} = \frac{1,5}{1,3} = 1, 5, \frac{5}{3}, \frac{1}{3}$

$\boxed{-5, -\frac{5}{3}, -1, -\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 5}$ test each

OR use factoring OR Quad formula

$3x^2-10x+5$ "Prime"

3: 1, 3 5: 1, 5

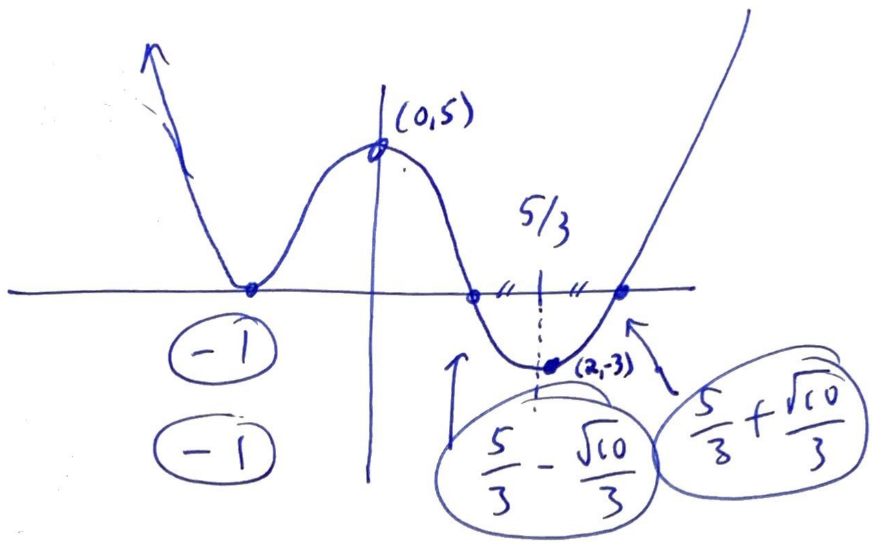
- So QuadFormula: $\frac{-(-10) \pm \sqrt{(-10)^2-4(3)(5)}}{2(3)}$

$= \frac{10 \pm \sqrt{100-60}}{6} = \frac{5}{3} \pm \frac{\sqrt{40}}{6} = \frac{5}{3} \pm \frac{\sqrt{10}}{3}$

$\frac{5}{3}+\frac{\sqrt{10}}{3}, \quad \frac{5}{3}-\frac{\sqrt{10}}{3}$

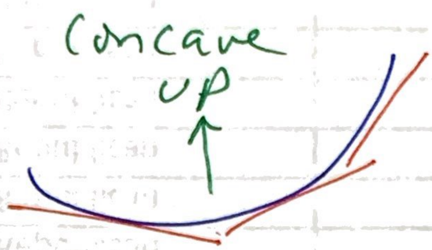
EX (cont.)

• Final Sketch

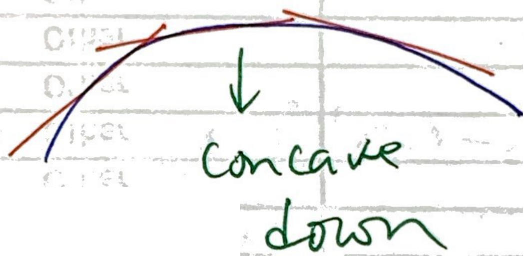


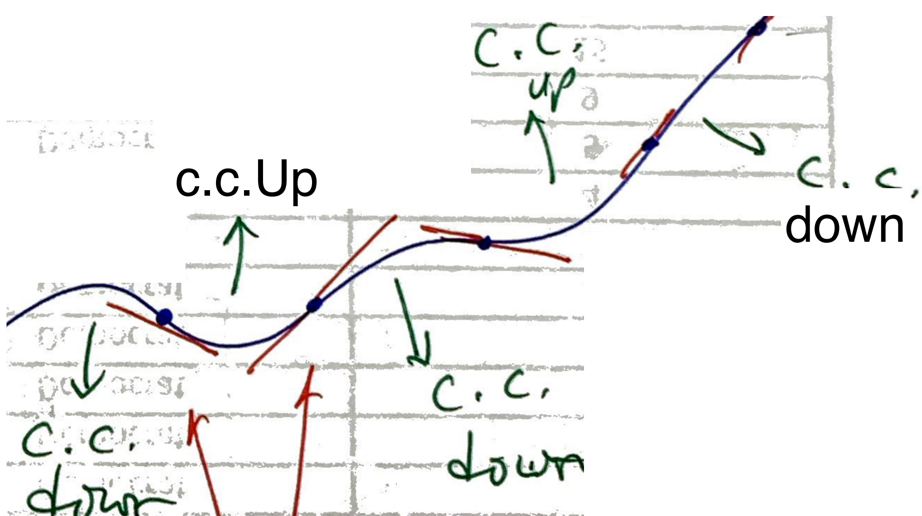
II The second derivative, f'', and shape of f

Def: a curve has a concave upward portion when the curve "cups" upwards



Def: a curve has a concave down portion when it "cups" down





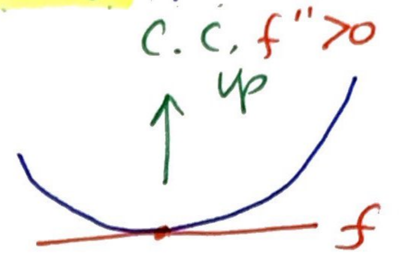
Def: Inflection points.

$f(x)$ changes concavity @ I.P.

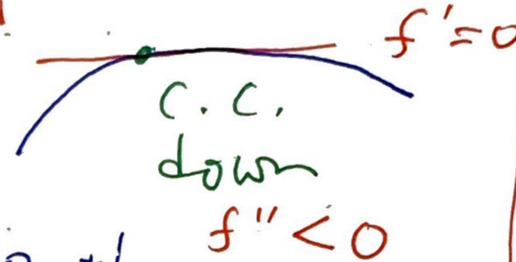
The Second derivative test

Suppose f'' is continuous at $x=c$

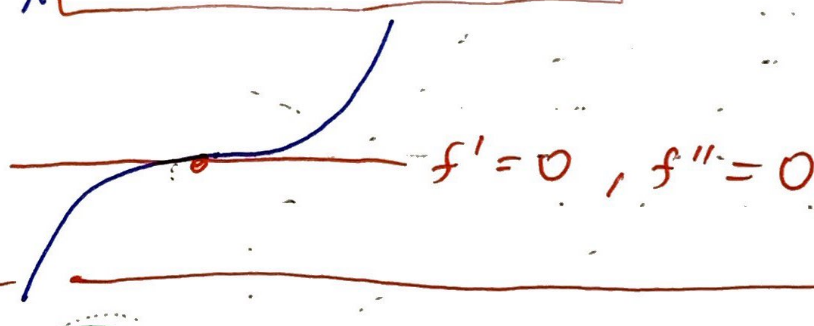
a) If $f'(c)=0$ and $f''(c)>0$ then "c" is a local $\boxed{\text{minimum}}$.



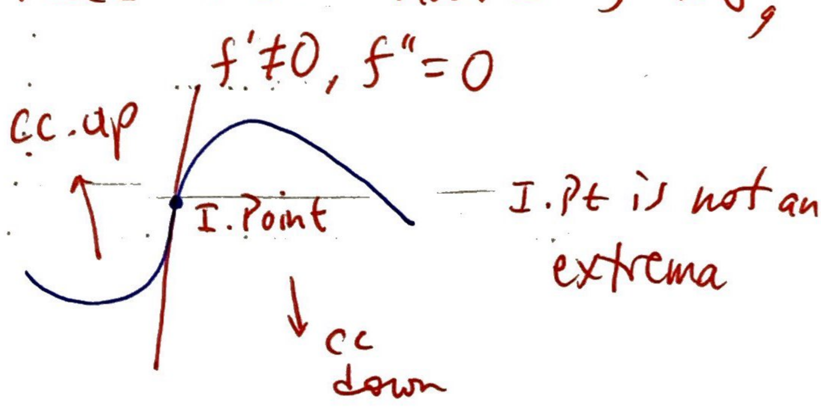
b) If $f'(c)=0$ and $f''(c)<0$ then "c" is a local $\boxed{\text{maximum}}$



c) If $f'(c)=0$ and $f''(c)=0$ the "c" is an horizontal $\boxed{\text{inflection point}}$ inconclusive



WARNING: $\boxed{\text{I. Pt}}$ need not have $f'=0$, only $\boxed{f''=0}$



EX Sketch $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$

Alt. form $f(x) = \frac{x^2+x-1}{x^2}$

(a) Vertical Asymptotes: (Denom $= 0$) @ $\boxed{x=0}$ VA

(b) Horizontal Asymptote: $\frac{\text{deg on top}}{\text{deg on bot.}}$

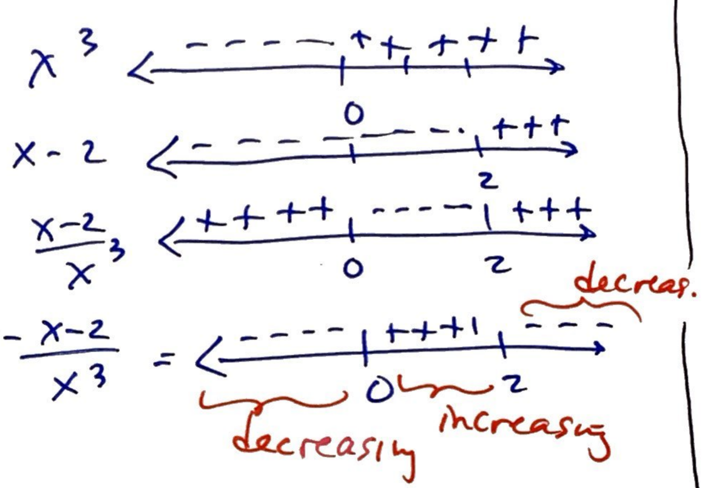
the $y_{\text{Hor. Asym}} = \frac{1}{1} = 1$ $\boxed{y=1}$ HA

(c) Increase/decrease:

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{-1}{x^3}(x-2)$$

line analysis

so

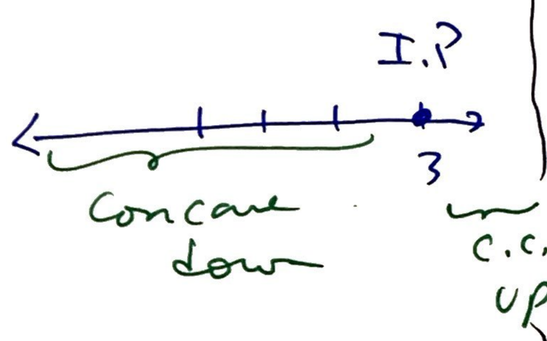


(d) Concavity

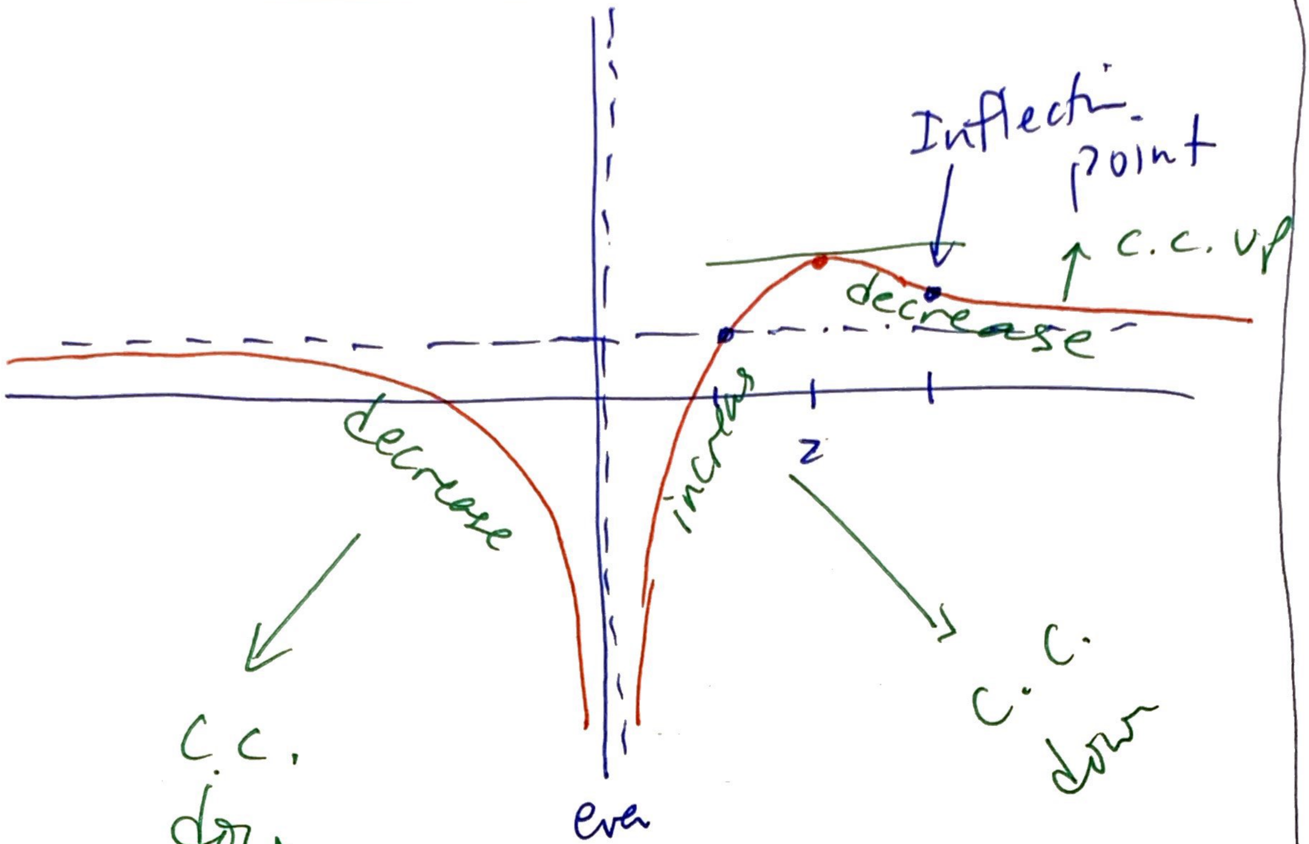
$$f'' = \left(-\frac{1}{x^2} + \frac{2}{x^3}\right)'$$

$$= \frac{2}{x^3} - \frac{6}{x^4}$$

$$= \frac{2}{x^4}(x-3)$$



(e) assemble sketch



• crossing HA? $\overset{\text{set}}{f(x) = 1}$

so $\dfrac{x^2 + x - 1}{x^2} = 1$

solve $\cancel{x^2} + x - 1 = \cancel{x^2}$

results: $\boxed{x = 1}$ is the only crossing of the HA $y = 1$

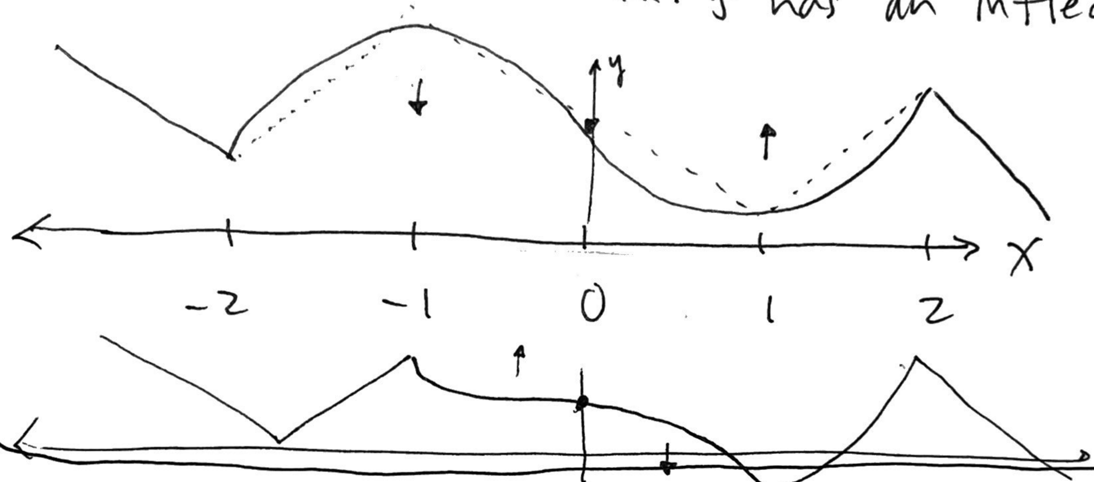
Ex Sketch a possible graph of a continuous function that satisfies

I. $f'<0$ in $x\in(-1,1)$ } i.e. $|x|<1$

II. $f'>0$ in $\underbrace{1<x<2 \text{ and } -2<x<-1}_{\text{i.e. } 1<|x|<2}$

III. $f'=-1$ for $x>2$ and for $x<-2$ } i.e. $|x|>2$

IV: f has an inflection point @ $(0,1)$



two possible graphs.